(D). *The probability function* $f(x_1, x_2, \cdots, x_n)$ *is continuous at every point* $x=(x_1, x_2, \cdots, x_n)$ *of the domain* T *of* P.

(E). *If* R *is any range of the domain* T *of* P *(the case in which* R *coincides with* T *not being excluded), the* n*-fold Lebesgue integral of* $f(x_1, x_2, \cdots, x_n)$ *exists over* R.[1]

While these postulates are not suggested as the best approach to a rigorous treatment of the problems of statistical probability, they serve to outline the general concept of a probability function in a way which leads very directly to the developments of the following sections. They therefore meet the specific requirements of the present paper.

4. *Fundamental Properties of the Probability Function*

Let $f(x_1, x_2, \cdots, x_n)$ be the probability function for a distribution of the set $(x_1, x_2, \cdots, x_n)$ of n attributes in a statistical population P, of domain T.

Any point $a \equiv (a_1, a_2, \cdots, a_n)$ of E_n is a vertex of an elementary rectangular region $(x_i, x_i+\delta x_i)$ consisting of all points of E_n of which the coordinates satisfy the inequalities $a_i<x_i<a_i+\delta x_i$, where the δx_i are positive and $i=1, 2, \cdots, n$. This region will be called a *cell* of E_n, and will be denoted by ΔX.

Let x be any point (except a frontier point) of T, and let the cell ΔX lie entirely within T. From the postulates laid down, we conclude at once that the typical relative frequency in ΔX is given, more and more closely as the n numbers δx_i all approach zero, by the product:

$$f(x_1, x_2, \cdots, x_n)\delta x_1 \delta x_2 \cdots \delta x_n. \tag{1}$$

Moreover, if H denotes T, or any subrange of T, it is readily seen that the typical relative frequency in H is given by the n-fold integral, extended over H:

$$\int\int \cdots \int f(x_1, x_2, \cdots, x_n)dx_1 dx_2 \cdots dx_n. \tag{2}$$

Since the typical relative frequency in T itself is manifestly unity, this integral, when extended over all of T, clearly has the value 1.

5. *Elementary Distribution Parameters*

Under the conditions of the first paragraph of the preceding section, let $h(x_1, x_2, \cdots, x_n)$ be any assigned function which is real-valued, single-valued, and continuous in T—and therewith also in P. Let S_q be a typical sample of P, consisting of q individuals. Let $x \equiv (x_1, x_2, \cdots, x_n)$ be any point (except a frontier point) of T, and let ΔX be a cell of E_n with a vertex at x.

If q is supposed to increase indefinitely, and the diameter of ΔX to approach zero (so that all of the δx_i approach zero), we may write, more and more accurately:

[1] So far as *bounded* and *closed* ranges in T are concerned, postulate E may be deduced, as a theorem, from postulate D.

$$qf(X)\Delta X = qf(x_1, x_2, \cdots, x_n)\delta x_1 \delta x_2 \cdots \delta x_n$$

$$= \text{the expected actual (not relative) frequency of } S_q \text{ in } \Delta X.$$

$$qh(X)f(X)\Delta X = qh(x_1, x_2, \cdots, x_n)f(x_1, x_2, \cdots, x_n)\delta x_1 \delta x_2 \cdots \delta x_n$$

$$= \text{the expected sum of the values assumed by } h \text{ for individuals of } S_q \text{ in } \Delta X.$$

Taking the n-fold integral of this expression over T, we obtain, more and more nearly as q increases, the expected sum of the values of $h(x_1, x_2, \cdots, x_n)$ for all individuals of S_q. Division by q then yields the typical arithmetic mean, $\bar{h}$, of the values of the function h:

$$(3) \qquad \bar{h} = \int\int \cdots \int h(x_1, x_2, \cdots, x_n)f(x_1, x_2, \cdots, x_n)dx_1 dx_2 \cdots dx_n.$$

Two very important arithmetic means calculated from a typical sample S_q are $\bar{x}_i$, the mean of the values of x_i; and $\lambda_t^{(i)}$, the mean of the values of $(x_i - \bar{x}_i)^t$, with $t = 2, 3, 4, \cdots$. These are:

$$(4) \qquad \bar{x}_i = (\sum x_i)/q,$$

and

$$(5) \qquad \lambda_t^{(i)} = \text{the } t\text{th centroidal moment of } x_i = \sum (x_i - \bar{x}_i)^t / q,$$

the summations being extended over all q individuals of the sample. In (4) and (5), i is not a summation index. The positive square root[1] of $\lambda_2^{(i)}$ is called the *standard deviation* of x_i, and is denoted by σ_i.

The *typical* values of (4) and (5)—the values which they may be expected to approach more and more closely as q is increased indefinitely—are found from formula (3) upon setting $h(x_1, x_2, \cdots, x_n)$ equal to x_i and to $(x_i - \bar{x}_i)^t$ respectively.[2]

A third basic parameter, denoted by r_{ij}, and called the *coefficient of correlation*[3] of the attributes measured by x_i and x_j is calculated from a typical sample S_q by means of the formula

$$(6) \qquad q\sigma_i\sigma_j r_{ij} = \sum (x_i - \bar{x}_i)(x_j - \bar{x}_j),$$

the summation being extended over all q individuals of the sample. In (6), i and j are not summation indices. The number of the parameters r_{ij} is n^2; but $r_{ji} = r_{ij}$, and $r_{ii} = 1$.

Since r_{ij} is plainly the arithmetic mean of the values, in S_q, of the function $(x_i - \bar{x}_i)(x_j - \bar{x}_j)/\sigma_i\sigma_j$, the *typical* value of r_{ij} is found from (3) upon substituting

[1] Clearly, $\lambda_2^{(i)} \geqq 0$. But, under our postulates, $\lambda_2^{(i)} \neq 0$. See the first footnote under the lemma, Section 7.

[2] The typical value of $\bar{x}_i$ is, of course, used in finding that of $\lambda_t^{(i)}$.

[3] This designation is, in certain respects, misleading. See Section 8.

this function for $h(x_1, x_2, \cdots, x_n)$, and using the typical values of σ_i and σ_j.

Let the variables be changed by a linear transformation of the special type $u_i = a_i x_i + b_i$, with non-vanishing determinant. It is readily seen that the means and centroidal moments of x_i and u_i are related by the formulas $\bar{u}_i = a_i \bar{x}_i + b_i$ and[1] $\lambda_t^{(u_1)} = a_i^t \lambda_t^{(x_1)}$. Moreover, the coefficient of correlation of x_i and x_j is equal to that of u_i and u_j, and is therefore an invariant.

Now subject the variables x_i and x_j, denoted by x and y, to the transformation $u = (x - \bar{x})/\sigma_x$, $v = (y - \bar{y})/\sigma_y$, and leave the remaining variables (if there are any) unchanged. By the remarks just made, $\bar{u} = \bar{v} = 0$, $\sigma_u = \sigma_v = 1$, and $r_{uv} = r_{xy} = r_{ij} = r$. Also, if the calculations are made from a sample S_q, we have $\sum u^2 = q\sigma_u^2 = q = \sum v^2 = q\sigma_v^2$, and $\sum uv = qr\sigma_u\sigma_v = qr$. We may therefore write the equations[2]

$$r = 1 - [\sum (u - v)^2]/2q \text{ and } r = -1 + [\sum (u + v)^2]/2q,$$

from which we conclude that $-1 \leqq r_{ij} \leqq 1$. Obviously, the *typical* r_{ij} must likewise lie on the interval $(-1, 1)$.[3]

Other basic parameters, which depend directly upon the r_{ij}, are: (1) the symmetric determinant $R = |r_{ij}|$, of nth order, having r_{ij} as the element in the ith row and the jth column; and (2) R_{ij}, the cofactor (the minor, with the proper sign) of the element r_{ij} in R.

The number of the parameters R_{ij} is n^2; but, of course, $R_{ji} = R_{ij}$. The typical values of R and of the R_{ij} are calculated from the typical values of the r_{ij}.

The probability function $f(x_1, x_2, \cdots, x_n)$ will be called *canonical* if the typical arithmetic mean value, $\bar{x}_i$, as determined by it, is equal to zero, $(i = 1, 2, \cdots, n)$. Any proposed probability function may be thrown into canonical form by a simple change of origin for the variables; that is, by a transformation of the type $u_i = x_i + b_i$, which leaves centroidal moments of order $\geqq 2$, as well as coefficients of correlation, invariant.

Example 1. Let the probability function be $f(x_1) = f(x) = Cx^2$, where C denotes a constant to be determined; and let the domain T be the open interval $0 < x < 4$, or the closed interval $0 \leqq x \leqq 4$, of E_1.

By Section 4, we have $\int_0^4 f(x)dx = 1$, whence $C = 3/64$. Therefore the typical values of $\bar{x}$ and $\sigma^2 (= \sigma_1^2)$ are:

$$\bar{x} = (3/64)\int_0^4 x^3 dx = 3,$$

$$\sigma^2 = (3/64)\int_0^4 (x - 3)^2 x^2 dx = 3/5.$$

Example 2. Let the probability function be $f(x_1, x_2) = Cx_1^2 e^{-(x_1^2 + x_2^2 + x_1 x_2)}$,

[1] $q\lambda_t^{(u_i)} = \sum (u_i - \bar{u}_i)^t$, in accordance with formula (5).

[2] These equations are taken from H. L. Rietz, *Mathematical Statistics*, p. 84.

[3] Under our postulates, the numerical value of the typical r_{ij} is less than 1 if $i \neq j$. See Section 8.

where C denotes a constant; and let the domain be the entire space E_2. From the formula $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2)dx_1dx_2 = 1$, we find[1] that $C = (3\sqrt{3})/(4\pi)$. Using this value of C, and noting[2] that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i f(x_1, x_2)dx_1dx_2 = 0$ for $i = 1, 2$, so that $f(x_1, x_2)$ is canonical in the domain E_2, we calculate the typical value of σ_1^2 as follows:

$$\sigma_1^2 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^4 e^{-(x_1^2+x_2^2+x_1x_2)}dx_1dx_2$$

$$= C\int_{-\infty}^{\infty}\left[x_1^4 e^{-x_1^2}\int_{-\infty}^{\infty} e^{-x_2(x_1+x_2)}dx_2\right]dx_1.$$

Substituting $x_2 = t - x_1/2$, $dx_2 = dt$, we find that the x_2-integral is equal to

$$e^{x_1^2/4}\int_{-\infty}^{\infty} e^{-t^2}dt = (\sqrt{\pi})e^{x_1^2/4}.$$

Hence

$$\sigma_1^2 = (C\sqrt{\pi})\int_{-\infty}^{\infty} x_1^4 e^{-3x_1^2/4}dx_1.$$

Integration by parts reduces this integral to that of Laplace (see the first footnote under the present example), and thus yields $\sigma_1^2 = (8\pi C)/(3\sqrt{3}) = 2$.

In like manner, we find the typical

$$\sigma_2^2 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 x_2^2 e^{-(x_1^2+x_2^2+x_1x_2)}dx_1dx_2 = C\int_{-\infty}^{\infty} x_1^2 e^{-x_1^2}Jdx_1,$$

where

$$J = \int_{-\infty}^{\infty} x_2^2 e^{-x_2(x_1+x_2)}dx_2.$$

---

[1] The n-fold integral ($n = 1, 2, 3, \cdots$):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}e^{-Q}dx_1dx_2\cdots dx_n,$$

extended throughout the space E_n, is readily evaluated by elementary methods whenever:

(1) Q is a positive-definite quadratic form in the n variables $x_1, x_2, \cdots, x_n$—see Section 10; and

(2) each of the exponents $m_1, m_2, \cdots, m_n$ is a positive integer or zero. If, in addition, all of the exponents $m_1, m_2, \cdots, m_n$ are equal to zero, the value of the integral is $(\pi^n/D)^{1/2}$, where D is the value of the discriminant (determinant) of Q. Thus, in particular, for $n = 1$, we have the important formula of Laplace:

$$\int_{-\infty}^{\infty} e^{-h^2x^2}dx = \sqrt{\pi}/h.$$

[2] The procedure suggested in the determination of σ_1^2 may be followed in evaluating the other integrals of the present example, and of later examples.

The substitution $x_2 = t - x_1/2$, $dx_2 = dt$, and reduction, yield

$$J = e^{x_1^2/4}[\sqrt{\pi}/2 + x_1^2\sqrt{\pi}/4].$$

Therefore we have

$$\sigma_2^2 = (C\sqrt{\pi}/2)\int_{-\infty}^{\infty} x_1^2 e^{-3x_1^2/4}dx_1 + (C\sqrt{\pi}/4)\int_{-\infty}^{\infty} x_1^4 e^{-3x_1^2/4}dx_1 = 1.$$

Finally, we calculate the typical

$$r_{12} = \frac{C}{\sigma_1\sigma_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^3 x_2 e^{-(x_1^2+x_2^2+x_1x_2)}dx_1dx_2 = -1/\sqrt{2}.$$

Example 3. Let the probability function be $f(x_1, x_2, x_3) = Ce^{-Q}$, where C is a constant, and

$$Q = 99x_1^2 + 75x_2^2 + 36x_3^2 - 170x_1x_2 - 116x_1x_3 + 100x_2x_3.$$

Let the domain be the entire space E_3.

The quadratic form Q, with discriminant

$$D = \begin{vmatrix} 99 & -85 & -58 \\ -85 & 75 & 50 \\ -58 & 50 & 36 \end{vmatrix},$$

is positive-definite,[1] and $D = 400$. Hence, applying the formula

$$C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-Q}dx_1dx_2dx_3 = 1,$$

and the footnote under Example 2, we find that $C = 20/(\pi^{3/2})$. Since $f(x_1, x_2, x_3)$ is obviously canonical in the domain E_3, the typical mean squared-deviations are

$$\sigma_i^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^2 f(x_1, x_2, x_3)dx_1dx_2dx_3 \qquad (i = 1, 2, 3);$$

and we find that $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$. Similarly, the typical coefficients of correlation are found from the formulas

$$\sigma_i\sigma_j r_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i x_j f(x_1, x_2, x_3)dx_1dx_2dx_3 \qquad (i, j = 1, 2, 3),$$

in which the typical values of σ_1, σ_2, σ_3 are employed. These formulas yield the typical values $r_{12} = 4/5$, $r_{13} = 1/2$, $r_{23} = -1/10$; and, of course, $r_{11} = r_{22} = r_{33} = 1$. Thus the typical basic determinant is

[1] See Section 10.

$$R = \begin{vmatrix} 1 & 4/5 & 1/2 \\ 4/5 & 1 & -1/10 \\ 1/2 & -1/10 & 1 \end{vmatrix}.$$

6. *Derived Probability Functions, and Array Distributions*[1]

Returning to the conditions stated in the first paragraph of Section 4, we now discuss certain probability functions expressing important aspects of the distribution there defined (for P, and in the domain T) by the function $f(x_1, x_2, \cdots, x_n)$.

Let that subset of P consisting of all individuals whose first k coordinates $(x_1, x_2, \cdots, x_k)$, $k<n$, correspond to points of the k-dimensional cell $a_i<x_i<a_i+\delta x_i$, $i=1; 2, \cdots, k$, with a vertex at the assigned point $(a_1, a_2, \cdots, a_k)$, be called an *array*, $P(12\cdots k)$, of P at the *pole* $(a_1, a_2, \cdots, a_k)$; and let the greatest of the k numbers δx_i, all of which are supposed positive, be called the *norm* of the array. That subset, $S_q(12\cdots k)$, of a typical sample S_q, consisting of all those individuals of S_q which belong to an array $P(12\cdots k)$, will be called a *sample array* of the same pole and norm.

Since the attributes measured by the n coordinates x_i may be rearranged in any desired order, no loss of generality is involved, for the above definitions, in the selection of the *first* k attributes.

It is readily seen that the probability function for the distribution of the set $(x_1, x_2, \cdots, x_k)$ of k attributes, when the other $n-k$ attributes are ignored, is

$$g(x_1, x_2, \cdots, x_k) = \int\int \cdots \int f(x_1, x_2, \cdots, x_n)dx_{k+1}dx_{k+2}\cdots dx_n,$$

the limits of the $(n-k)$-fold integral being determined, in the familiar manner, by the frontier of the range (domain) T. That is to say, the relative frequency, in a typical sample S_q, of those individuals which belong to an array $S_q(12\cdots k)$, of norm ζ and pole $(x_1, x_2, \cdots, x_k)$, may be expected to be given, more and more nearly as q is increased indefinitely and ζ approaches zero, by $g(x_1, x_2, \cdots, x_k)\,\delta x_1\,\delta x_2, \cdots, \delta x_k$. Thus, if an individual is selected at random from S_q, the *probability* that the individual will belong to the array $S_q(12\cdots k)$ becomes more and more nearly equal to $g(x_1, x_2, \cdots, x_k)\,\delta x_1\,\delta x_2, \cdots, \delta x_k$.

Now let an individual N be selected at random from the sample array $S_q(12\cdots k)$ of norm ζ and pole $(x_1, x_2, \cdots, x_k)$. Let the probability function for the distribution, *within the population array* $P(12\cdots k)$, of the set $(x_{k+1}, x_{k+2}, \cdots, x_n)$ of $n-k$ attributes be, more and more nearly as the norm ζ of that array approaches zero[2], $r(x_1, x_2, \cdots, x_n)$. Then the probability that the last $n-k$ coordinates of N will be interior to specified intervals $(x_i, x_i+\delta x_i)$,

[1] See footnote to the heading of Section 8.

[2] The variables fixing the array have assigned values; hence r is defined, *throughout the array*, as a function of the $n-k$ variables $x_{k+1}, x_{k+2}, \cdots, x_n$.

$i=k+1, k+2, \cdots, n$, is given, more and more nearly as ζ and the $n-k$ numbers δx_i all approach zero, by the product $r(x_1, x_2, \cdots, x_n)\ \delta x_{k+1}\ \delta x_{k+2} \cdots \delta x_n$.

It follows that if an individual is selected at random from S_q, the probability that its coordinates will be interior to the intervals $(x_i, x_i+\delta x_i)$, $i=1, 2, \cdots, n$, is given, more and more nearly as q is increased indefinitely and all of the n numbers δx_i approach zero, by the product $g(x_1, x_2, \cdots, x_k)\ r(x_1, x_2, \cdots, x_n)\ \delta x_1\ \delta x_2 \cdots \delta x_n$. But, by hypothesis, $f(x_1, x_2, \cdots, x_n)$ is the probability function for the distribution of the set $(x_1, x_2, \cdots, x_n)$ of all n attributes in P; and therefore the last-mentioned probability is also given, more and more nearly, by the product $f(x_1, x_2, \cdots, x_n)\ \delta x_1\ \delta x_2 \cdots \delta x_n$. We thus find that

$$r(x_1, x_2, \cdots, x_n) = \frac{f(x_1, x_2, \cdots, x_n)}{g(x_1, x_2, \cdots, x_k)}$$

is the distribution function for the set $(x_{k+1}, x_{k+2}, \cdots, x_n)$ of attributes within the array $P(12 \cdots k)$, where $x_1, x_2, \cdots, x_k$ have assigned values.

Example 4. Returning to the distribution defined, in Example 3, by the function $f(x_1, x_2, x_3)$ in the domain E_3, we derive several probability functions:

(1) $g(x_1)$ = the distribution function for x_1 when x_2 and x_3 are ignored

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_2dx_3 = (2/\sqrt{2\pi})e^{-2x_1^2}$$

(2) $r(x_1, x_2, x_3)$ = the distribution function for the pair (x_2, x_3) in an array $P(1)$ where x_1 is assigned

$$= \frac{f(x_1, x_2, x_3)}{g(x_1)}$$

$$= (10\sqrt{2}/\pi)e^{-(97x_1^2+75x_2^2+36x_3^2-170x_1x_2-116x_1x_3+100x_2x_3)}$$

(3) $t(x_2, x_3)$ = the distribution function for (x_2, x_3) when x_1 is ignored

$$= \int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1$$

$$= (20\sqrt{99}/99\pi)e^{-(40/99)(5x_2^2+x_2x_3+5x_3^2)}$$

(4) $u(x_1, x_2, x_3)$ = the distribution function for x_1 in an array $P(23)$, where x_2 and x_3 have assigned values

$$= \frac{f(x_1, x_2, x_3)}{t(x_2, x_3)}$$

$$= (\sqrt{99}/\sqrt{\pi})e^{-Z},$$

with $Z = 99x_1^2 + 7225x_2^2/99 + 3364x_3^2/99 - 170x_1x_2 - 116x_1x_3 + 9860x_2x_3/99$.

(5) $y(x_2, x_3)$ = the distribution function for x_2 when x_3 has an assigned value and x_1 is ignored

$$= \frac{t(x_2, x_3)}{\int_{-\infty}^{\infty} t(x_2, x_3)dx_2} = (\sqrt{\pi}/\sqrt{2})e^{2x_3^2}t(x_2, x_3)$$

$$= (20/\sqrt{198\pi})e^{-(1/99)(200x_2^2+40x_2x_3+2x_3^2)}.$$

Since these are all *probability* functions, their integrals, extended through the respective domains, must be equal to unity. The reader may verify that:

$$\int_{-\infty}^{\infty} g(x_1)dx_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(x_1, x_2, x_3)dx_2dx_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t(x_2, x_3)dx_2dx_3$$

$$= \int_{-\infty}^{\infty} u(x_1, x_2, x_3)dx_1 = \int_{-\infty}^{\infty} y(x_2, x_3)dx_2 = 1.$$

A number of other probability functions may obviously be derived, in the same manner, from the distribution defined in Example 3.

7. *Regression Equations, and Correlation*

The conditions of the first paragraph of Section 4 are again assumed. Thus the probability function for the distribution of x_1 in an array $P(23 \cdots n)$ is

$$H(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n) \Big/ \int f(x_1, x_2, \cdots, x_n)dx_1,$$

the limits of the integral being determined, of course, by the frontier of T. The *typical* arithmetic mean, $\bar{x}_1(23 \cdots n)$, of the values assumed by x_1 *within the array* $P(23 \cdots n)$—that is, the expected mean of x_1 in the array $S_q(23 \cdots n)$ of a typical sample S_q with sufficiently large q—is therefore given, more and more nearly as the norm of the array approaches zero, by

$$\bar{x}_1(23 \cdots n) = \int x_1 H(x_1, x_2, \cdots, x_n)dx_1$$

$$= \int x_1 f(x_1, x_2, \cdots, x_n)dx_1 \Big/ \int f(x_1, x_2, \cdots, x_n)dx_1.$$

In the language of statistics, $x_1 = \bar{x}_1(23 \cdots n) = p(x_2, x_3, \cdots, x_n)$ is called the *regression equation* of x_1 on the set $(x_2, x_3, \cdots, x_n)$ of $n-1$ attributes; and the locus of this equation, in E_n, is called the *regression hypersurface* of x_1 on $(x_2, x_3, \cdots, x_n)$. When p is a first-degree polynomial in its $n-1$ variables, the regression is *linear*, and the regression hypersurface is a regression *hyperplane*.[1]

[1] Hypersurfaces (hyperplanes) are ordinary surfaces (planes) for $n=3$, and curves (straight lines) for $n=2$.

The formula $x_1 = \bar{x}_1(23 \cdots n)$ is one of *typical* regression; if q increases indefinitely, the corresponding regression, as found from S_q by elementary methods, may be expected to be given more and more closely by this formula.

The degree of *correlation* of the attribute x_1 with the set of $n-1$ attributes $(x_2, x_3, \cdots, x_n)$ is defined, when $n>2$, to be the degree of concentration of the individuals of P about the regression hypersurface $x_1 = \bar{x}_1(23 \cdots n)$. When $n=2$, there is a certain concentration about each of the regression curves $x_1 = \bar{x}_1(2)$, $x_2 = \bar{x}_2(1)$; and then, by definition, the greater of these two concentrations fixes the correlation of x_1 with x_2.

Let $w(x_2, x_3, \cdots, x_n)$ be any single-valued and continuous function of the $n-1$ arguments indicated. The typical mean squared-deviation of x_1 away from its corresponding values on the hypersurface $x_1 = w$ is given by the n-fold integral, extended over T:

$$(7) \qquad \int\int \cdots \int [x_1 - w(x_2, x_3, \cdots, x_n)]^2 f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n.$$

Lemma. This integral assumes its minimum (and smallest possible) value, necessarily positive,[1] when $w = \bar{x}_1(23 \cdots n) = p(x_2, x_3, \cdots, x_n)$.

For, integrating first with respect to x_1, the values of the other variables being fixed, we obtain $K(w) = \int (x_1 - w)^2 f dx_1$, where the parameter w, and the limits of integration,[2] are constants depending upon the values of $x_2, x_3, \cdots, x_n$. It is evident that the function $K(w)$ assumes a minimum (which is also the smallest possible) value. This value occurs when $K'(w) = 0$; that is, when $\int x_1 f dx_1 = w \int f dx_1$, or $w = \bar{x}_1(23 \cdots n)$. The lemma follows immediately.

In particular, the integral (7) yields: (1) the typical σ_1^2, when w is equal to the constant $\bar{x}_1$ (typical); and (2) the typical mean squared-deviation, $\sigma^2_{1,23\cdots n}$, of x_1 away from the regression hypersurface $x_1 = \bar{x}_1(23 \cdots n)$, when $w = \bar{x}_1(23 \cdots n)$.

The positive square root, $\sigma_{1,23\cdots n}$, of $\sigma^2_{1,23\cdots n}$, is a convenient distribution parameter. In fact, the ratio $(\sigma_{1,23\cdots n})/\sigma_1$ furnishes a measure of the degree to which the statistics (individuals of P) are *scattered away from the regression hypersurface* $x_1 = \bar{x}_1(23 \cdots n)$. This ratio is >0 and $\leqq 1$, as shown by the lemma.

A commonly employed direct measure of the *concentration of the statistics* about the regression hypersurface $x_1 = \bar{x}_1(23 \cdots n)$, and accordingly a *measure of the correlation* existing between the attribute x_1 and the set $(x_2, x_3, \cdots, x_n)$ of $n-1$ attributes, is provided by the non-negative square root, $V_{1,23\cdots n}$, of the non-negative number $V^2_{1,23\cdots n} = 1 - (\sigma^2_{1,23\cdots n})/\sigma_1^2$. The parameter $V_{1,23\cdots n}$ is called the *correlation ratio* of x_1 on the set of attributes $(x_2, x_3, \cdots, x_n)$. Ob-

[1] f, being a probability function, is non-negative in its domain T, so that (7) is non-negative. And if (7) were equal to zero, T would not be a *range* in E_n, as required by our postulates, but would lie on a hypersurface.

[2] The integral may be extended over more than one interval of the line corresponding to the fixed values of $x_2, x_3, \cdots, x_n$ if this line cuts the frontier of T in more than two points.

viously, $0 \leqq V_{1,23\ldots n} < 1$; and the correlation between x_1 and $(x_2, x_3, \cdots, x_n)$ rises from non-existence toward perfection as $V_{1,23\ldots n}$ increases from 0 toward 1.

Example 5. For the distribution given in Example 3, we find that:

$$\bar{x}_1(23) = \int_{-\infty}^{\infty} x_1 f(x_1, x_2, x_3)dx_1 \Big/ \int_{-\infty}^{\infty} f(x_1, x_2, x_3)dx_1 = (85x_2 + 58x_3)/99,$$

$$\bar{x}_2(13) = \int_{-\infty}^{\infty} x_2 f dx_2 \Big/ \int_{-\infty}^{\infty} f dx_2 = (17x_1 - 10x_3)/15,$$

$$\bar{x}_3(12) = \int_{-\infty}^{\infty} x_3 f dx_3 \Big/ \int_{-\infty}^{\infty} f dx_3 = (29x_1 - 25x_2)/18.$$

Hence the regression of each attribute on the other two is linear, and the regression planes are $x_1 = (85x_2 + 58x_3)/99$, $x_2 = (17x_1 - 10x_3)/15$, and $x_3 = (29x_1 - 25x_2)/18$. We next find that:

$$\sigma^2_{1,23} = \iiint [x_1 - \bar{x}_1(23)]^2 f(x_1, x_2, x_3)dx_1dx_2dx_3 = 1/198 = 0.00505,$$

$$\sigma^2_{2,13} = \iiint [x_1 - \bar{x}_2(13)]^2 f dx_1dx_2dx_3 = 1/150 = 0.00667,$$

$$\sigma^2_{3,12} = \iiint [x_1 - \bar{x}_3(12)]^2 f dx_1dx_2dx_3 = 1/72 = 0.01389,$$

the integration being extended from $-\infty$ to $+\infty$ with respect to each variable. Very high correlation exists between each attribute and the remaining pair of attributes. In fact, recalling (from Example 3) that $\sigma_1 = \sigma_2 = \sigma_3 = 1/2$, we find that: $V_{1,23} = 0.9898$, $V_{2,13} = 0.9864$, $V_{3,12} = 0.9718$.

Example 6. For the distribution given in Example 2, only one of the regression functions $\bar{x}_1(2)$, $\bar{x}_2(1)$ is linear. It will be found, in fact, that the regression equation of x_2 on x_1 is:

$$x_2 = \bar{x}_2(1) = \int_{-\infty}^{\infty} x_1^2 x_2 e^{-(x_1^2 + x_2^2 + x_1x_2)}dx_2 \Big/ \int_{-\infty}^{\infty} x_1^2 e^{-(x_1^2 + x_2^2 + x_1x_2)}dx_2$$

$$= \int_{-\infty}^{\infty} x_2 e^{-x_2(x_1 + x_2)}dx_2 \Big/ \int_{-\infty}^{\infty} e^{-x_2(x_1 + x_2)}dx_2 = -x_1/2\ ;$$

while that of x_1 on x_2 is:

$$x_1 = \bar{x}_1(2) = \int_{-\infty}^{\infty} x_1^3 e^{-(x_1^2 + x_2^2 + x_1x_2)}dx_1 \Big/ \int_{-\infty}^{\infty} x_1^2 e^{-(x_1 + x_2^2 + x_1x_2)}dx_1$$

$$= -(6x_2 + x_2^3)/(4 + 2x_2^2).$$

The calculation of the typical values of $\sigma_{1,2}$, $\sigma_{2,1}$, $V_{1,2}$, $V_{2,1}$ is left to the reader.

8. *Best-Fitting Hyperplanes, and Coefficients of Multiple Correlation*[1]

When appreciable correlation exists between the attribute x_1 and the set of $n-1$ attributes $(x_2, x_3, \cdots, x_n)$, it may often be exhibited, and even measured with much accuracy, by a parameter which is more easily calculated (especially from a sample S_q) than the correlation ratio $V_{1,23\cdots n}$.

The conditions of the first paragraph of Section 4 are again assumed. It is also assumed, merely for convenience and with no loss of generality, that $f(x_1, x_2, \cdots, x_n)$ is in *canonical* form, so that the typical $\bar{x}_i$ equals zero, $(i=1, 2, \cdots, n)$. Finally, it is assumed that R_{11}, of the basic determinant R of the distribution, is not equal to zero.

Then, in the class of all *linear* functions of $(x_2, x_3, \cdots, x_n)$, there exists one and only one,

$$b_{12}x_2 + b_{13}x_3 + \cdots + b_{1n}x_n + c \tag{8}$$

which, when substituted for $w(x_2, x_3, \cdots, x_n)$ in (7), yields, for that integral, a smaller value than any other linear w. The coefficients of this unique linear function are

$$c = 0, \qquad b_{1j} = -(\sigma_1 R_{1j})/(\sigma_j R_{11}) ; \tag{9}$$

and

$$x_1 = -(\sigma_1/R_{11}) \sum_{j=2}^{n} (R_{1j}x_j/\sigma_j) \tag{10}$$

will be called the "best-fitting" x_1-hyperplane[2] on $(x_2, x_3, \cdots, x_n)$.

In fact, let the n-fold integral

$$L(b_{12}, b_{13}, \cdots, b_{1n}, c) = \int\int \cdots \int \left(x_1 - c - \sum_{j=2}^{n} b_{1j}x_j\right)^2 f dx_1 dx_2 \cdots dx_n,$$

in which $f = f(x_1, x_2, \cdots, x_n)$, be extended over T. Equating the partial derivative dL/dc to zero, and indicating n-fold integration by one symbol, with differentials omitted, we obtain:[3]

$$\int (x_1 - b_{12}x_2 - b_{13}x_3 - \cdots - b_{1n}x_n - c)f = 0,$$

or $\bar{x}_1 - b_{12}\bar{x}_2 - b_{13}\bar{x}_3 - \cdots - b_{1n}\bar{x}_n = c = 0.$

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[1] Here, as in other sections, when a particular attribute, or set of attributes, is distinguished for special analysis, this is done merely for notational convenience, and no sacrifice of generality is implied.

[2] "Best fit" is here defined in the sense of the principle of least squares.

[3] Of course, $\int f = 1$; and, since f is canonical, we have the further relations:

$$\bar{x}_j = \int x_j f = 0, \quad \sigma_i\sigma_j r_{ij} = \int x_i x_j f \qquad (i, j = 1, 2, \cdots, n).$$

Setting $c=0$, we now write the $n-1$ equations $dL/db_{1j}=0$:

$$\int (x_1 - b_{12}x_2 - b_{13}x_3 - \cdots - b_{1n}x_n)x_j f = 0.$$

Cancelling the factor σ_j, we thus have:

$$b_{12}r_{2j}\sigma_2 + b_{13}r_{3j}\sigma_3 + \cdots + b_{1n}r_{nj}\sigma_n = r_{1j}\sigma_1 \qquad (j = 2, 3, \cdots, n). \tag{11}$$

It is recognized at once that the system (11) of linear equations uniquely specifies the numbers b_{1j}, and with the values given in (9), when the determinant of the coefficients of these numbers does not vanish.[1] But that determinant is plainly the product of $(\sigma_2\sigma_3 \cdots \sigma_n)$ and R_{11}, which was assumed to be different from zero.

The typical dispersion of the statistics away from the hyperplane (10) is the positive square root, $\bar{\sigma}_{1,23\ldots n}$, of the n-fold integral over T:

$$\begin{aligned}\bar{\sigma}^2_{1,23\cdots n} &= \int \left[x_1 + (\sigma_1/R_{11}) \sum_{j=2}^{n} (R_{1j}x_j/\sigma_j) \right]^2 f \\ &= (\sigma_1^2/R_{11}^2) \int \left[\sum_{j=1}^{n} (R_{1j}x_j/\sigma_j) \right]^2 f\end{aligned}$$

which is merely (7) with w set equal to the right-hand member of (10).[2] We may write:

$$\begin{aligned}\bar{\sigma}^2_{1,23\cdots n} &= (\sigma_1^2/R_{11}^2) \Big[\sum_{j=1}^{n} R_{1j}^2 + 2 \sum_{j=2}^{n} R_{11}R_{1j}r_{1j} + 2 \sum_{j=3}^{n} R_{12}R_{1j}r_{2j} \\ &\qquad + \cdots + 2R_{1,n-1}R_{1n}r_{n-1,n} \Big] \\ &= (\sigma_1^2/R_{11}^2) \sum_{j=1}^{n} R_{1j} \Big(R_{1j} + \sum_{i=1}^{n\,\prime} R_{1i}r_{ji} \Big),\end{aligned} \tag{12}$$

the (primed) i-summation omitting the term with $i=j$. By an elementary property of determinants, the parenthesis in (12) is equal to R or to 0 according as $j=1$ or $j\neq 1$. Hence $\bar{\sigma}^2_{1,23\ldots n}=\sigma_1^2(R/R_{11})$, and

$$\bar{\sigma}_{1,23\cdots n} = \sigma_1(R/R_{11})^{1/2}. \tag{13}$$

By the lemma, and the footnote just made:

$$\sigma_{1,23\cdots n} \leqq \bar{\sigma}_{1,23\cdots n} \leqq \sigma_1. \tag{14}$$

[1] See Cramer's Rule, in elementary algebra, for the solution of a system of linear equations with non-vanishing determinant.

[2] By the determination of the minimizing linear function (10), $\bar{\sigma}^2_{1,23\cdots n} \leqq \sigma_1^2$. Moreover, since σ_1^2 is given by (7) with w equal to $\bar{x}_1(=0)$, the inequality sign prevails unless $R_{1j}=0$ for $j=2, 3, \cdots, n$.

The non-negative square root, $r_{1,23\cdots n}$, of the non-negative number

$$r^2_{1,23\cdots n} = 1 - (\bar{\sigma}^2_{1,23\cdots n})/\sigma_1^2 \tag{15}$$

serves as a direct measure of the degree of concentration of the statistics about the best-fitting hyperplane (10), this concentration rising from non-existence toward perfection as $r_{1,23\cdots n}$ increases[1] from 0 toward 1.

When $n=2$, we have, by (13), $\bar{\sigma}^2_{1,2} = \sigma_1^2(1-r_{12}^2)$, whence

$$r_{12}^2 = 1 - (\bar{\sigma}^2_{1,2}/\sigma_1^2) = r^2_{1,2} \tag{15a}$$

so that the (non-negative) parameter $r_{1,2}$ is numerically equal to r_{12}. When $n>2$, the parameter $r_{1,23\cdots n}$ is called the *coefficient of multiple correlation* of x_1 with the set $(x_2, x_3, \cdots, x_n)$ of $n-1$ attributes. That the designation is a little misleading appears in the paragraph above, where the exact nature of this parameter is stated. It is evident that very high correlation might exist—very high concentration of the statistics about the regression hypersurface $x_1 = \bar{x}_1(23\cdots n)$—with $r_{1,23\cdots n}$ very small; but if $r_{1,23\cdots n}$ is large (nearly equal to 1), we are assured of high correlation between x_1 and $(x_2, x_3, \cdots, x_n)$. The suitability of $r_{1,23\cdots n}$ as a measure of this correlation is fully exhibited in the formula

$$r_{1,23\cdots n} \leqq V_{1,23\cdots n} \tag{16}$$

which follows immediately from (14).

By the lemma, Section 7, the regression hypersurface $x_1 = \bar{x}_1(23\cdots n)$ is the "best-fitting" surface of the type $x_1 = w(x_2, x_3, \cdots, x_n)$, where w is a single-valued and continuous function in T. It therefore coincides, when the regression of x_1 on $(x_2, x_3, \cdots, x_n)$ is *linear*, with the hyperplane (10); and in that case, of course, $\bar{\sigma}_{1,23\cdots n} = \sigma_{1,23\cdots n}$, $r_{1,23\cdots n} = V_{1,23\cdots n}$.

Example 7. For the distribution of Example 3, the regression of each attribute on the remaining pair of attributes was found to be linear (see Example 5). These regressions may therefore be checked by (10), with $n=3$, and by the two corresponding equations

$$x_2 = -(\sigma_2/R_{22}) \sum_{j=1,3} (R_{2j}x_j/\sigma_j), \quad x_3 = -(\sigma_3/R_{33}) \sum_{j=1,2} (R_{3j}x_j/\sigma_j),$$

if we set

$$\sigma_1 = \sigma_2 = \sigma_3 = 1/2, \quad R = 0.02, \quad R_{11} = 0.99, \quad R_{22} = 0.75,$$

$$R_{33} = 0.36, \quad R_{12} = R_{21} = -0.85, \quad R_{13} = R_{31} = -0.58, \quad R_{23} = R_{32} = 0.50.$$

Similarly, the typical values of $\sigma^2_{1,23}$, $\sigma^2_{2,13}$, $\sigma^2_{3,12}$, as obtained in Example 5, may be checked by (13) and the two corresponding formulas: $\bar{\sigma}^2_{2,13} = \sigma_2^2(R/R_{22})$, $\bar{\sigma}^2_{3,12} = \sigma_3^2(R/R_{33})$. The coefficients of multiple correlation are equal, in this case,

[1] It is obvious that $0 \leqq r_{1,23\cdots n} < 1$.

to the correlation ratios calculated in Example 5. For the distribution of (x_2, x_3), with x_1 ignored, the function $t(x_2, x_3)$, given in Example 4, shows that there is linear regression of x_2 on x_3; in fact, we find that

$$\int_{-\infty}^{\infty} x_2 t(x_2, x_3) dx_2 \Big/ \int_{-\infty}^{\infty} t(x_2, x_3) dx_2 = -x_3/10.$$

By formula (10), the regression equation $x_2 = -x_3/10$ ought to be the same as $x_2 = -(\sigma_2/Q_{22})(Q_{23}x_3/\sigma_3)$, where

$$Q = \begin{vmatrix} r_{12} & r_{23} \\ r_{32} & r_{33} \end{vmatrix} = \begin{vmatrix} 1 & -1/10 \\ -1/10 & 1 \end{vmatrix}$$

is the basic determinant of the distribution considered, and Q_{ij} denotes the cofactor of the element r_{ij} in Q—so that $Q_{22} = r_{33} = 1$, $Q_{23} = -r_{32} = 1/10$. Such is found to be the case. In the same manner, if i and j denote any two of the three indices 1, 2, 3, in either order, we find that the regression of x_i on x_j, with the third attribute ignored, is linear; and that its equation, which reduces to $x_i = (r_{ij}\sigma_i x_j)/\sigma_j$, is given by the formula $x_i = -(\sigma_i/Q_{ii})\ (Q_{ij}x_j/\sigma_j)$, where

$$Q = \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix}, Q_{ii} = r_{jj} = 1, Q_{ij} = -r_{ji} = -r_{ij}.$$

Example 8. For the distribution discussed in Examples 2 and 6, we found that the regression of x_2 on x_1 was linear. Hence the regression equation $x_2 = \bar{x}_2(1) = -x_1/2$ should be the same as that of the best-fitting x_2-line on x_1, given by formula (10): $x_2 = -\sigma_2 R_{21} x_1/\sigma_1 R_{22}$. Recalling that $\sigma_1^2 = 2$, $\sigma_2^2 = 1$, $r_{12} = -1/\sqrt{2}$,

$$R = \begin{vmatrix} 1 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1 \end{vmatrix}, R_{21} = 1/\sqrt{2}, R_{22} = 1,$$

we obtain the desired check.

The best-fitting x_1-line on x_2 is $x_1 = -\sigma_1 R_{12} x_2/\sigma_2 R_{11} = -x_2$. It is not, however, the regression curve (see Example 6) of x_1 on x_2; and therefore $r_{1,2}$ (= the numerical value of $r_{12} = 1/\sqrt{2} = 0.707$) does not measure the full extent of the correlation of x_1 with x_2. In this distribution, we have $V_{2,1} = r_{2,1} = r_{1,2} = 1/\sqrt{2}$; but $V_{1,2} > r_{1,2}$, and $V_{1,2}$ may perhaps be much more nearly equal to 1 than is indicated by the value of $r_{1,2}$.

9. *Array Parameters, and Coefficients of Partial Correlation*[1]

Under the conditions of the first paragraph of Section 4, we have $g(x_1, x_2, \cdots, x_k) = \int\int \cdots \int dx_{k+1} dx_{k+2} \cdots dx_n$ as the distribution function for $(x_1, x_2, \cdots, x_k)$ with $x_{k+1}, x_{k+2}, \cdots, x_n$ ignored; and $U(x_1, x_2, \cdots, x_n) = f/g$ as that for $(x_{k+1}, x_{k+2}, \cdots, x_n)$ within the array $P(12 \cdots k)$, where $x_1, x_2, \cdots, x_k$ have fixed values. It is understood that k denotes any integer $\geqq 1$ and $< n$.

[1] See the footnote to the heading of Section 8.

Employing $U(x_1, x_2, \cdots, x_n)$, in which the values of $x_1, x_2, \cdots, x_k$ are assigned, just as we have hitherto employed $f(x_1, x_2, \cdots, x_n)$, we may easily define, *for the distribution of* $(x_{k+1}, x_{k+2}, \cdots, x_n)$ *within the array* $P(12 \cdots k)$, all parameters, regression equations, best-fitting hyperplanes, etc., previously set up for the distribution of $(x_1, x_2, \cdots, x_n)$ in P. We have only to observe that if $h(x_1, x_2, \cdots, x_n)$ is any single-valued and continuous function in T, the typical arithmetic mean of the values assumed by h within $P(12 \cdots k)$ is given by the $(n-k)$-fold integral

$$(17) \qquad \int\int \cdots \int h(x_1, x_2, \cdots, x_n)U(x_1, x_2, \cdots, x_n)dx_{k+1}dx_{k+2} \cdots dx_n,$$

and is thus a function[1] of the assigned values of $x_1, x_2, \cdots, x_k$.

Let a typical sample S_q be partitioned into a (finite) set of arrays of the type $S_q(12 \cdots k)$, and let the arithmetic mean of h be calculated for the individuals of each sample array. Then let the arithmetic mean of these array-means be determined for the entire partition of S_q. The *typical* value of the resulting parameter—the value which it may be expected to approach more and more closely as q increases indefinitely, and the norms of the partition arrays in S_q all approach zero—is clearly the familiar

$$\bar{h} = \int\int \cdots \int h(x_1, x_2, \cdots, x_n)f(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n$$

of Section 5. Analytical verification is immediate. The typical arithmetic mean $f(h; 12 \cdots k)$, of the typical arithmetic array-mean of h given in (17) is expressed by a k-fold integral of an $(n-k)$-fold integral, namely by

$$\int\int \cdots \int \Big[\int\int \cdots \int hUdx_{k+1}dx_{k+2} \cdots dx_n\Big] gdx_1dx_2 \cdots dx_k\,;$$

and, recalling that g is independent of the variables $x_{k+1}, x_{k+2}, \cdots, x_n$, while $U=f/g$, we see that

$$(18) \qquad f(h\,;12 \cdots k) = \int\int \cdots \int hfdx_1dx_2 \cdots dx_n.$$

Letting i, j denote integers of the set $k+1, k+2, \cdots, n$, we signalize three especially interesting and important cases:

(*A*) $\bar{x}_i(12 \cdots k)$ = the typical arithmetic array-mean found, from (17), for $h=x_i$

(*B*) $\sigma_i^2(12 \cdots k)$ = that for $h = [x_i - \bar{x}_i(12 \cdots k)]^2$

(*C*) $r_{ij}(12 \cdots k)$ = that for $h = \dfrac{[x_i - \bar{x}_i(12 \cdots k)][x_j - \bar{x}_j(12 \cdots k)]}{\sigma_i(12 \cdots k)\sigma_j(12 \cdots k)}$,

[1] See below for certain cases in which these array means are known to be independent of the values of the k variables fixing the array.

where $\sigma_i(12 \cdots k)$ is the positive square root of the array parameter (B).

Evidently, (B) is the typical square of the standard deviation of x_i within the array $P(12 \cdots k)$; and (C) is the typical coefficient of correlation of x_i and x_j within that array. By (18), the typical arithmetic means of these two typical array parameters are:

$$(19)\quad \overset{2}{\sigma}_{i,12\cdots k} = \int\int \cdots \int [x_i - \bar{x}_i(12 \cdots k)]^2 f dx_1 dx_2 \cdots dx_n,$$

$$(20)\quad r_{ij,12\cdots k} = \int\int \cdots \int \frac{[x_i - \bar{x}_i(12 \cdots k)][x_j - \bar{x}_j(12 \cdots k)]}{\sigma_i(12 \cdots k)\sigma_j(12 \cdots k)} f dx_1 dx_2 \cdots dx_n.$$

The typical arithmetic mean of the typical $\bar{x}_i(12 \cdots k)$ is plainly $\bar{x}_i$.

The parameter $r_{ij,12\cdots k}$ is the typical *coefficient of partial correlation* of x_i and x_j, in the distribution of $(x_1, x_2, \cdots, x_n)$, when the measures of $x_1, x_2, \cdots, x_k$ are assigned, and $n-k-2$ measures are ignored. This parameter is certainly of high statistical significance in *normal* distributions of $(x_1, x_2, \cdots, x_n)$, defined in Section 10, and in distributions approximating normality. Normal distributions possess several outstanding properties, among which are the following two:

(1) If $1 \leqq k \leqq n-1$, the regression of any attribute on any set of k other attributes, with $n-k-1$ attributes ignored, is linear.

(2) If $1 \leqq k \leqq n-1$, the typical array parameter $\sigma_i(12 \cdots k)$ is a constant, and is therefore independent of the values of $x_1, x_2, \cdots, x_k$ fixing the array; and if $1 \leqq k \leqq n-2$, the typical array parameter $r_{ij}(12 \cdots k)$ is likewise a constant. Hence the typical $\sigma_i(12 \cdots k)$ is equal to the positive square root, $\sigma_{i,12\cdots k}$, of the typical $\sigma^2_{i,12\cdots k}$; and the typical $r_{ij}(12 \cdots k)$ and $r_{ij,12\cdots k}$ are equal.

For any distribution, whether normal or not, with the properties (1) and (2), the typical $\sigma_{i,12\cdots k}$ and $r_{ij,12\cdots k}$, and therewith also the typical $\sigma_i(12 \cdots k)$ and $r_{ij}(12 \cdots k)$, may be calculated directly from the elementary parameters σ_i and R. We will now develop these important formulas.

Suppose for convenience, in accordance with the footnote to the heading of Section 8, that we seek the typical $r_{12,ab\cdots c}$, the coefficient of partial correlation of x_1 and x_2, for assigned values of k other variables of which the indices, arranged in natural order, are $a, b, \cdots, c$; and that the indices of the $n-k-2$ ignored variables, arranged in natural order, are $s, t, \cdots, u$. We must first find the typical $\sigma_{1,ab\cdots c}$ and $\sigma_{2,ab\cdots c}$.

Let $r_{a\varepsilon}, r_{\beta\zeta}, \cdots, r_{\vartheta\theta}$ be any $m < n$ elements of the basic determinant R for the distribution of $(x_1, x_2, \cdots, x_n)$; and let no two of these m elements stand in the same row or in the same column of R. Deleting, from R, the m rows and m columns in which these elements stand, we denote by $R_{a\beta\cdots\delta,\varepsilon\zeta\cdots\theta}$ the resulting determinant,[1] which is of order $n-m$. Then the distribution, D, of $(x_1, x_a, x_b, \cdots, x_c)$, with $x_2, x_s, x_t, \cdots, x_u$ ignored, has the basic determinant

[1] In particular, R_{ij} equals $\pm R_{i,j}$, according as $i+j$ is even or odd.

$Q=R_{2st\cdots u,2st\cdots u}$, of which the order is $k+1$. For if i and j are integers of the set $1, a, b, \cdots, c$, we see that r_{ij} has the same typical value[1] whether defined by the probability function f in the full distribution, F, of $(x_1, x_2, \cdots, x_n)$ or by the probability function $G(x_1, x_a, x_b, \cdots, x_c)=\int\int\int \cdots \int\int dx_2dx_sdx_t \cdots dx_u$ in the distribution D.

As stated above, F is assumed to have the properties (1) and (2). It is also assumed, for convenience, that f is canonical in F—so that G is obviously canonical in D.

The arguments of Section 8, applied to F with its distribution function f and its determinant R, may now be applied verbatim to D with its distribution function G and its determinant Q. It is thus found, in D, that the best-fitting linear formula for x_1 on $(x_a, x_b, \cdots, x_c)$ is[2]

$$x_1 = -(\sigma_1/Q_{11})\sum(Q_{1j}x_j/\sigma_j), \tag{21}$$

with j assuming, in the summation, the k values $a, b, \cdots, c$. By reason of the assumed linear regression, (21) is also the regression equation of x_1 on $(x_a, x_b, \cdots, x_c)$. The typical mean squared-deviation of x_1 away from the values given by (21) is the $(k+1)$-fold integral:

$$\bar{\sigma}^2_{1,ab\cdots c} = \int\int\int \cdots \int\left[x_1+(\sigma_1/Q_{11})\sum(Q_{1j}x_j/\sigma_j)\right]^2 G\,dx_1dx_adx_b\cdots dx_c,$$

which may be expressed as the n-fold integral (over T):

$$\bar{\sigma}^2_{1,ab\cdots c} = \int\int \cdots \int\left[x_1+(\sigma_1/Q_{11})\sum(Q_{1j}x_j/\sigma_j)\right]^2 f\,dx_1dx_2\cdots dx_n. \tag{22}$$

But the linear regression identifies (22) with the integral (over T):

$$\int\int \cdots \int [x_1-\bar{x}_1(ab\cdots c)]^2 f\,dx_1dx_2\cdots dx_n,$$

and consequently, by the general formula (19), with the typical $\sigma^2_{1,ab\cdots c}$.

Hence we have only to calculate $\bar{\sigma}^2_{1,ab\cdots c}$ by (22), proceeding exactly as in Section 8 (where $k=n-1$), except that R is replaced by Q. We find:

$$\bar{\sigma}_{1,ab\cdots c} = \sigma_1(Q/Q_{11})^{1/2} = \sigma_1(R_{2st\cdots u,2st\cdots u}/R_{12st\cdots u,12st\cdots u})^{1/2}. \tag{23}$$

In like manner, the distribution of $(x_2, x_a, x_b, \cdots, x_c)$, with disregard of $x_1, x_s, x_t, \cdots, x_u$, has the determinant $W=R_{1st\cdots u,1st\cdots u}$, and the probability function $\int\int\int \cdots \int\int dx_1dx_sdx_t \cdots dx_u$; it yields the regression equation:[3]

$$x_2 = \bar{x}_2(ab\cdots c) = -(\sigma_2/W_{22})\sum(W_{2j}x_j/\sigma_j), \tag{24}$$

[1] Similarly, σ_i has the same value in both distributions.

[2] Q_{ij} is the cofactor of the element r_{ij} in Q, $(i, j=1, a, b, \cdots, c)$; but r_{ij} does not necessarily stand in the i-th row and j-th column of Q.

[3] W_{ij} is the cofactor of the element r_{ij} in W, $(i, j=2, a, b, \cdots, c)$. In the summation, j assumes the k values $a, b, \cdots, c$.

and the typical value:

$$\bar{\sigma}_{2,ab\cdots c} = \sigma_2(R_{1st\cdots u,1st\cdots u}/R_{12st\cdots u,12st\cdots u})^{1/2}. \tag{25}$$

From the general formula (20), we finally obtain:

$$\sigma_{1,ab\cdots c}\sigma_{2,ab\cdots c}r_{12,ab\cdots c}$$

$$= \int [x_1 + (\sigma_1/Q_{11}) \sum (Q_{1j}x_j/\sigma_j)]\ [x_2 + (\sigma_2/W_{22}) \sum (W_{2j}x_j/\sigma_j)]f$$

where an n-fold integral, extended over T, is denoted on the right by a single symbol, and differentials are omitted.[1] In the summations, j assumes the k values $a, b, \cdots, c$.

The simplification of this result is straightforward. It is necessary only to recall that, since f is canonical, the expressions $x_i^2 f$ and $x_i x_j f$ yield respectively, when integrated over T, σ_i^2 and $\sigma_i\sigma_j r_{ij}$; that $Q = R_{2st\cdots u,2st\cdots u}$ and $W = R_{1st\cdots u,\ 1st\cdots u}$; and that $\sigma_{1,ab\cdots c}$ and $\sigma_{2,ab\cdots c}$ are given by (23), (25). We find the formula:

$$r_{12,ab\cdots c} = \frac{R_{1st\cdots u,2st\cdots u}}{[(R_{1st\cdots u,1st\cdots u})(R_{2st\cdots u,2st\cdots u})]^{1/2}} \tag{26}$$

which becomes, when $k = n-2$, simply

$$r_{12,34\cdots n} = \frac{R_{1,2}}{(R_{1,1}R_{2,2})^{1/2}} = \frac{-R_{12}}{(R_{11}R_{22})^{1/2}}.$$

In the derivation of (26), it was assumed that F possessed properties (1) and (2). It is evidently sufficient, however, that, *for the particular set of k variables* $x_a, x_b, \cdots, x_c$ *chosen*, the regressions of x_1 and x_2 on this set of variables be linear, and that the parameters $\sigma_1(ab\cdots c)$ and $\sigma_2(ab\cdots c)$ be constants. In this case, the coefficient of correlation of x_1 and x_2 has the same value, that given by (26), in every array $P(ab\cdots c)$ of these k variables. If, in addition, the regression of x_1 on x_2, and that of x_2 on x_1, are linear when the other $n-2$ variables are ignored—and thus linear also within the array $P(ab\cdots c)$—formula (26) yields the actual *correlation ratio* of x_1 on x_2, as well as that of x_2 on x_1, for every array $P(ab\cdots c)$. In such cases, the coefficient of partial correlation $r_{12,ab\cdots c}$ serves as a measure of the correlation still existing between the attributes x_1 and x_2 when the possible effect of variation of the k attributes $x_a, x_b, \cdots, x_c$ has been eliminated.

Obviously,

$$r_{12,ab\cdots c} = r_{21,ab\cdots c}. \tag{27}$$

Whether or not the regression of x_1 on the set $(x_a, x_b, \cdots, x_c)$ of k variables

[1] The assumption is here made that the parameters $\sigma_1(ab\cdots c)$, $\sigma_2(ab\cdots c)$ are *constants*, so that they are equal, respectively, to the parameters $\sigma_{1,ab\cdots c}$, $\sigma_{2,ab\cdots c}$.

is linear, the reader will understand the significance of the parameters

$$V^2_{1,ab\cdots c} = 1 - (\sigma^2_{1,ab\cdots c})/\sigma^2_1, \tag{28}$$

and

$$r^2_{1,ab\cdots c} = 1 - (\bar{\sigma}^2_{1,ab\cdots c})/\sigma^2_1, \tag{29}$$

and of their non-negative square roots $V_{1,ab\cdots c}$, $r_{1,ab\cdots c}$. These parameters were discussed in Sections 7 and 8, for the case $k=n-1$. When the regression mentioned is linear, we have, of course, $r_{1,ab\cdots c}=V_{1,ab\cdots c}$; and, in any case,

$$r_{1,ab\cdots c} \leqq V_{1,ab\cdots c}. \tag{30}$$

Example 9. Returning to the distribution analyzed in Examples 3, 4, 5, 7, we find, in accordance with (4) of Example 4, and (1) of Example 5, that

$$\sigma^2_1(23) = \int_{-\infty}^{\infty} [x_1 - \bar{x}_1(23)]^2 u(x_1, x_2, x_3) dx_1 = 1/198.$$

This parameter has here a value independent of x_2 and x_3, and is therefore equal to the $\sigma_{1,2^3}{}^2$ calculated in Example 5. Similarly, it will be found that $\sigma_2{}^2(13)=\sigma_{2,13}{}^2$ and $\sigma_3{}^2(12)=\sigma_{3,12}{}^2$.

The typical arithmetic means of x_2 and x_3 in an array $P(1)$ are (see Example 7):

$$\bar{x}_2(1) = -(\sigma_2/Q_{22})(Q_{21}x_1/\sigma_1),$$

with

$$Q = \begin{vmatrix} r_{22} & r_{21} \\ r_{12} & r_{11} \end{vmatrix},$$

and

$$\bar{x}_3(1) = -(\sigma_3/W_{33})(W_{31}x_1/\sigma_1),$$

with

$$W = \begin{vmatrix} r_{33} & r_{31} \\ r_{13} & r_{11} \end{vmatrix}.$$

Thus we obtain the two typical values:

$$\sigma^2_2(1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x_2 - \bar{x}_2(1)]^2 r(x_1, x_2, x_3) dx_2 dx_3$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x_2 - 4x_1/5]^2 r\, dx_2 dx_3,$$

$$\sigma_3^2(1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_3 - \bar{x}_3(1)]^2 r(x_1, x_2, x_3)dx_2dx_3$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_3 - x_1/2]^2 r\, dx_2dx_3,$$

where $r(x_1, x_2, x_3)$ is the function (2) of Example 4. These integrals yield the values $\sigma_2^2(1) = 9/100$ and $\sigma_3^2(1) = 3/16$, which are independent of the value of x_1 fixing the array $P(1)$. Hence $\sigma_{2,1}^2 = 9/100$ and $\sigma_{3,1}^2 = 3/16$.

Further, the typical $r_{23}(1)$ is given by the formula:

$$\sigma_2(1)\sigma_3(1)r_{23}(1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[x_2 - 4x_1/5][x_3 - x_1/2]r(x_1, x_2, x_3)dx_2dx_3.$$

Substituting $\sigma_2(1) = 3/10$, $\sigma_3(1) = \sqrt{3}/4$, and evaluating the integral, we find that $r_{23}(1) = -5\sqrt{3}/9$, a result which is independent of x_1. Thus $r_{23,1} = -5\sqrt{3}/9 = -0.9623$, and a much stronger correlation than that indicated by $r_{23} = -1/10$ (see Example 3) is shown to exist between x_2 and x_3 when the influence of x_1 has been eliminated. Since conditions are here satisfied for the application of the formula[1]

$$r_{23,1} = R_{2,3}/(R_{2,2}R_{3,3})^{1/2} = -R_{23}/(R_{22}R_{33})^{1/2},$$

with

$$R = \begin{vmatrix} r_{22} & r_{23} & r_{21} \\ r_{32} & r_{33} & r_{31} \\ r_{12} & r_{13} & r_{11} \end{vmatrix},$$

the reader may check the value of $r_{23,1}$.

In the same manner, it may be shown that $r_{13}(2) = r_{13,2} = 29/3\sqrt{9} = 0.972$, and $r_{12}(3) = r_{12,3} = 17/\sqrt{297} = 0.986$.

10. *Normal Distribution*

The general quadratic n-ary form is

$$A = \sum_{i,j=1}^{n} a_{ij}x_ix_j,$$

with $a_{ji} = a_{ij}$; and it may be noted that $2a_{ij} = \partial^2 A/\partial x_i\partial x_j$. Thus the unary form is $a_{11}x_1^2$, the binary is $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$, and the ternary is $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$. The *discriminant* of the form A is the determinant of n-th order:

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[1] The arrangement of R corresponds to the order of the subscripts in $r_{22,1}$.

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

For $h=1, 2, \cdots, n$, let $D(h)$ denote any determinant, of order h, obtained from D by striking out $n-h$ rows, and $n-h$ columns with the same indices. When regarded as a determinant of first order, each element a_{ij} of the principal diagonal of D is a $D(1)$; and D is itself the only $D(n)$. There are $n!/h!(n-h)!$ determinants of the type $D(h)$, $h=1, 2, \cdots, n$; and each is called a "principal minor," of order h, in D. In particular, the principal minors of order h standing in the upper-left-hand and lower-right-hand corners of D will be denoted by ${}^{(h)}D$ and $D_{(h)}$ respectively.

The quadratic form A, of discriminant D, is called *positive-definite* if (1) all the coefficients a_{ij} are real numbers, and (2) the value of A is positive whenever the n variables x_j have real values which are not all equal to zero. If a positive-definite form A is converted, by a non-singular[1] linear transformation

$$x_i = \sum_{j=1}^{n} b_{ij} y_j \qquad (i = 1, 2, \cdots, n),$$

with real coefficients b_{ij}, into $c_1y_1^2+c_2y_2^2+\cdots+c_ny_n^2$, then $c_j>0$ for $j=1, 2, \cdots, n$.

In order that the form A, with real coefficients, be positive-definite, it is necessary and sufficient that, for $h=1, 2, \cdots, n$, every $D(h)$ be greater than zero. This condition is, however, satisfied if, for $h=1, 2, \cdots, n$, the n inequalities ${}^{(h)}D>0$, or the equivalent n inequalities $D_{(h)}>0$, are valid.

Let the form A, with discriminant D, be positive-definite in the set of n variables $(x_1, x_2, \cdots, x_n)$. It is evident that the terms of second degree in any k of these variables, $(1 \leqq k \leqq n-1)$, constitute a quadratic form which is positive-definite in the k variables. Moreover, let fixed (constant) values be assigned to any k variables, say to $x_1, x_2, \cdots, x_k$. Then A becomes a quadratic polynomial, Q, in the $n-k$ variables $x_{k+1}, x_{k+2}, \cdots, x_n$. It may be written $Q(x_{k+1}, x_{k+2}, \cdots, x_n)=E+F+G$, where E is the quadratic form $\sum a_{ij}x_ix_j$, $(i, j=k+1, k+2, \cdots, n)$, F is the linear form $2\sum b_jx_j$, $(j=k+1, k+2, \cdots, n)$, and $G=\sum a_{ij}x_ix_j$, with $(i, j=1, 2, \cdots k)$, is a constant. The constant b_j is related to the constants $x_1, x_2, \cdots, x_k$ by the formula

$$b_j = \sum_{i=1}^{k} a_{ji}x_i, \qquad (j = k+1, k+2, \cdots, n).$$

The form E being positive-definite in $x_{k+1}, x_{k+2}, \cdots, x_n$, its discriminant, $D_{(n-k)}$, is not equal to zero, and therefore the system of $n-k$ linear equations

[1] The n-th order determinant of the b_{ij} is not equal to zero.

$$b_j + \sum_{i=k+1}^{n} a_{ji}x_i = 0, \qquad (j = k+1, k+2, \cdots, n),$$

has a unique solution: $x_{k+1}=c_1$, $x_{k+2}=c_2$, $\cdots$, $x_n=c_n$. Thus A may be represented, when $x_1, x_2, \cdots, x_k$ have constant values, as the sum of a constant, J, dependent upon the values of these k variables, and a positive-definite quadratic form in the variables $u_h=x_h-c_h$, $(h=k+1, k+2, \cdots, n)$:

$$A = J + \sum_{i,j=k+1}^{n} a_{ij}(x_i - c_i)(x_j - c_j).$$

For the cases $n=2$, $n=3$, this representation of A is familiar in elementary analytic geometry.

A distribution of the set $(x_1, x_2, \cdots, x_n)$ of n attributes is called "normal" if and only if:

(1) the range of each x_j, $(j=1, 2, \cdots, n)$, is from $-\infty$ to $+\infty$; and
(2) the probability function for the distribution can be written

$$f(x_1, x_2, \cdots, x_n) = C\, e^{-\Sigma a_{ij}(x_i-c_i)(x_j-c_j)} = C\, e^{-Q},$$

where C is a suitably determined constant, and Q is a positive-definite quadratic form in $(u_1, u_2, \cdots, u_n)$, with $a_{ji}=a_{ij}$ and $u_h=x_h-c_h$. In accordance with footnote to Example 2, we readily find that

$$1 = C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-\Sigma a_{ij}u_iu_j}du_1du_2\cdots du_n = C(\pi^n/D)^{1/2},$$

whence $C=(D/\pi^n)^{1/2}$, where D is the discriminant of the form Q. For this normal distribution of $(x_1, x_2, \cdots, x_n)$, it is clear that the typical value of $\bar{x}_j$ is $\bar{x}_j=c_j$, $j=1, 2, \cdots, n$. When all the c_j are equal to zero, the distribution function is canonical.

Professor Karl Pearson has shown[1] that the probability function for a normal distribution of $(x_1, x_2, \cdots, x_n)$ in which the basic determinant of the r_{ij} is R, and the standard deviation of x_j is σ_j, may be expressed:

$$f(x_1, x_2, \cdots, x_n) = Me^{-\Sigma(R_{ij}u_iu_j/\sigma_i\sigma_j)/(2R)},$$

with $u_h=x_h-c_h$, $(h=1, 2, \cdots, n)$, and $M=1/[(2\pi)^{n/2}\sigma_1\sigma_2\cdots\sigma_n\sqrt{R}]$. The summation in the exponent of e is, of course, a positive-definite quadratic form in $u_1, u_2, \cdots, u_n$, the discriminant of which has $R_{ij}/\sigma_i\sigma_j$ as element in the i-th row and the j-th column. Denoting by K the determinant of n-th order having simply R_{ij} as element in the i-th row and the j-th column, we see that ${}^{(h)}K>0$ and $K_{(h)}>0$, $(h=1, 2, \cdots, n)$.

It follows that the coefficients of correlation r_{ij} in a normal distribution are bound by certain inequalities. In fact:[2]

[1] See Whittaker and Robinson, *The Calculus of Observations*, pages 340–342.

[2] See Bôcher, *Introduction to Higher Algebra*, pages 30–33.

$$K = R^{n-1} \text{ and } K_{(h)} = {}^{(n-h)}RR^{h-1}, \quad h = 1, 2, \cdots, n-1.$$

The last of these equations shows that $R>0$ when n is even. Setting $h=n-1$, and noting that ${}^{(1)}R=r_{11}=1$, we obtain $K_{(n-1)}=R^{n-2}$ and conclude that $R>0$ when n is odd. Thus, in any case, ${}^{(n-h)}R>0$, $h=1, 2, \cdots, n-1$. Therefore ${}^{(h)}R>0, h=1, 2, \cdots, n$; and consequently $R(h)>0, h=1, 2, \cdots, n$, where $R(h)$ denotes any principal minor, of order h, in R.

Using the canonical Pearsonian expression of the normal function,

$$f(x_1, x_2, \cdots, x_n) = Me^{-\Sigma(R_{ij}x_ix_j/\sigma_i\sigma_j)/(2R)},$$

we easily find that

$$\int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n)dx_1$$

is

$$Me^{-q\Sigma h_{ij}R_{ij}x_ix_j}\int_{-\infty}^{\infty} e^{-qx_1(h_{11}R_{11}x_1+2h_{12}R_{12}x_2+\cdots+2h_{1n}R_{1n}x_n)}dx_1,$$

where $q=1/2R$, $h_{ij}=1/\sigma_i\sigma_j$, $h_{jj}=1/\sigma_j^2$, and the summation indices i, j, in the power of e outside the integration sign, range *from* 2 *to* n. A change of the integration variable from x_1 to

$$t = x_1 + (\sigma_1/R_{11})(R_{12}x_2/\sigma_2 + \cdots + R_{1n}x_n/\sigma_n),$$

with $dx_1=dt$, leads at once to the following expression for the integral

$$\int_{-\infty}^{\infty} f\,dx_1 = (M\sigma_1\sqrt{2\pi R}/\sqrt{R_{11}})e^{-q\Sigma h_{ij}R_{ij}x_ix_j}e^{(R_{12}x_2/\sigma_2+\cdots+R_{1n}x_n/\sigma_n)^2/(2RR_{11})}$$
$$= \left\{1/\left[(2\pi)^{(n-1)/2}\sigma_2\sigma_3\cdots\sigma_n\sqrt{R_{11}}\right]\right\}e^{-\{\Sigma(R_{ij}x_ix_j/\sigma_i\sigma_j)-V^2/R_{11}\}/(2R)},$$

where $V=R_{12}x_2/\sigma_2+\cdots+R_{1n}x_n/\sigma_n$, and the summation indices i, j range from 2 to n.

Denoting by T the determinant $R_{(n-1)}=R_{11}$, and by T_{ij} the cofactor, in T, of the element r_{ij}—which stands in row $i-1$ and column $j-1$ of T—, we quickly find that this result may be written[1]

$$\int_{-\infty}^{\infty} f(x_1, x_2, \cdots, x_n)dx_1 = \left\{1/\left[(2\pi)^{(n-1)/2}\sigma_2\sigma_3\cdots\sigma_n\sqrt{T}\right]\right\}e^{-\Sigma(T_{ij}x_ix_j/\sigma_i\sigma_j)/(2T)}$$

with $i, j=2, 3, \cdots, n$.

Thus, in a normal distribution, N, of $(x_1, x_2, \cdots, x_n)$, with basic determinant R, the distribution of $(x_2, x_3, \cdots, x_n)$, when x_1 is ignored, is likewise

[1] See, again, Bôcher, pages 30–33. Note that:

$$\begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} = RR_{(n-2)} = RT_{22}, \qquad \begin{vmatrix} R_{11} & R_{13} \\ R_{21} & R_{23} \end{vmatrix} = RT_{23}, \text{ etc.}$$

normal, and has $R_{(n-1)}$ as basic determinant; and the standard deviation of x_j, $j=2, 3, \cdots, n$, is the same in both distributions. The conclusion is immediate that the distribution of any k attributes, $(1 \leqq k \leqq n-1)$, when the remaining $n-k$ are disregarded, is normal, and has a k-rowed minor of R as basic determinant; and that the standard deviation of each attribute in the distribution is the same as in N.

Let us now consider the distribution of $(x_{k+1}, x_{k+2}, \cdots, x_n)$ within an array of N where $x_1, x_2, \cdots, x_k$ have assigned values. According to the developments of Section 9, the probability function for this array distribution is

$$U(x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)/g(x_1, x_2, \cdots, x_k)$$

where f is the probability function, expressed in Pearsonian and (for convenience) canonical form, for the normal distribution N, and g is the distribution function for $(x_1, x_2, \cdots, x_k)$ when the remaining $n-k$ attributes are ignored.

By the theorem just established, we readily see that

$$g(x_1, x_2, \cdots, x_k) = \left\{1/\left[(2\pi)^{k/2}\sigma_1\sigma_2 \cdots \sigma_k\sqrt{H}\right]\right\} e^{-\Sigma(H_{ij}x_ix_j/\sigma_i\sigma_j)/(2H)},$$

where $H = {}^{(k)}R$, and the indices in the summation, which is a positive-definite quadratic form in $x_1, x_2, \cdots, x_k$, range from 1 to k. Hence $U(x_1, x_2, \cdots, x_n) = Ce^{-W}$ if

$$C = 1/\left[(2\pi)^{(n-k)/2}\sigma_{k+1}\sigma_{k+2} \cdots \sigma_n\sqrt{R/H}\right] \tag{1}$$

and

$$W = \left\{1/(2R)\right\} \sum_{i,j=1}^{n} (R_{ij}x_ix_j/\sigma_i\sigma_j) - \left\{1/2H\right\} \sum_{i,j=1}^{k} (H_{ij}x_ix_j/\sigma_i\sigma_j) \tag{2}$$

$$= \left\{1/(2R)\right\} \sum_{i,j=1}^{k} (R_{ij}x_ix_j/\sigma_i\sigma_j) - \left\{1/2H\right\} \sum_{i,j=1}^{k} (H_{ij}x_ix_j/\sigma_i\sigma_j) + Q$$

with Q expressible, as already remarked above, in the form

$$Q = \left\{1/(2R)\right\}\left\{ \sum_{i,j=k+1}^{n} (R_{ij}x_ix_j/\sigma_i\sigma_j) + 2\sum_{j=k+1}^{n} (x_j/\sigma_j)(R_{j1}x_1/\sigma_1 + \cdots + R_{jk}x_k/\sigma_k)\right\}$$

$$= \left\{1/(2R)\right\}\left\{ \sum_{i,j=k+1}^{n} (R_{ij}u_iu_j/\sigma_i\sigma_j) + h(x_1, x_2 \cdots, x_k)\right\},$$

where $u_r = x_r - c_r$, $r = k+1, k+2, \cdots, n$, c_r is a homogeneous linear function of $x_1, x_2, \cdots, x_k$, and h is a quadratic polynomial in these k variables.

It is obvious that the quadratic form

$$\sum_{i,j=k+1}^{n} (R_{ij}u_iu_j/\sigma_i\sigma_j)$$

is positive-definite in the $n-k$ variables $u_{k+1}, u_{k+2}, \cdots, u_n$; and it is readily found that $x_r - c_r = 0$ is the familiar best fitting hyperplane of x_r on $(x_1, x_2, \cdots, x_k)$, with the remaining $n-k-1$ variables ignored.

Treating $x_1, x_2 \cdots, x_k$ as constants, we find that

$$W = L(x_1, x_2, \cdots, x_k) + \{1/(2R)\} \sum_{i,j=k+1}^{n} (R_{ij}u_iu_j/\sigma_i\sigma_j),$$

where the value of the constant $L(x_1, x_2, \cdots, x_k)$ depends, of course, upon the (constant) values assigned to $x_1, x_2, \cdots, x_k$.

Since U is the probability function for the distribution of $(x_{k+1}, x_{k+2}, \cdots, x_n)$ when $x_1, x_2, \cdots, x_k$ have assigned values, we may write:

$$C \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-W} dx_{k+1} dx_{k+2} \cdots dx_n = 1,$$

and therefore:

$$(1/C)e^{L(x_1, x_2, \cdots, x_k)} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\Sigma(R_{ij}u_iu_j/\sigma_i\sigma_j)/(2R)}$$

$$= \sigma_{k+1}\sigma_{k+2} \cdots \sigma_n(\pi^n/D)^{1/2},$$

where D is the determinant of order $n-k$ obtained upon dividing each element of $R_{(n-k)}$ by $1/2R$. Hence $L(x_1, x_2, \cdots, x_k)$ is independent of the values assigned to $x_1, x_2, \cdots, x_k$; and, in fact, simple calculation shows that $L = 0$.

It follows that U is the function:

$$U = Ce^{-\Sigma(R_{ij}u_iu_j/\sigma_i\sigma_j)/(2R)},$$

$i, j = k+1, k+2, \cdots, n$, and therewith that *the distribution of* $(x_{k+1}, x_{k+2}, \cdots, x_n)$, *for assigned values of* $x_1, x_2, \cdots, x_k$, *is normal.*

We conclude at once that c_h, in $u_h = x_h - c_h$, is the typical mean of x_h, $h = k+1, k+2, \cdots, n$, in an array $P(12 \cdots k)$ where $x_1, x_2, \cdots, x_k$ have assigned values.

Since the distribution of $(x_{k+1}, x_{k+2}, \cdots, x_n)$ in an array $P(12 \cdots k)$ is now known to be normal, we may write its probability function, in Pearsonian form, as follows:

$$U = Ke^{-\Sigma(Z_{ij}p_ip_j/t_it_j)/(2Z)},$$

where

$$t_j = \sigma_j(12 \cdots k), \quad p_j = x_j - \bar{x}_j(12 \cdots k), \quad K = 1/[(2\pi)^{(n-k)/2}t_{k+1}t_{k+2} \cdots t_n\sqrt{Z}],$$

and

$$Z = \begin{vmatrix} r_{aa}(12\cdots k) & r_{ab}(12\cdots k) & \cdots & r_{ac}(12\cdots k) \\ r_{ba}(12\cdots k) & r_{bb}(12\cdots k) & \cdots & r_{bc}(12\cdots k) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ r_{ca}(12\cdots k) & r_{cb}(12\cdots k) & \cdots & r_{cc}(12\cdots k) \end{vmatrix},$$

with $a=k+1$, $b=k+2, \cdots, c=n$, $r_{ij}(12\cdots k)=r_{ji}(12\cdots k)$, $r_{jj}(12\cdots k)=1$ $(i, j=k+1, k+2, \cdots, n)$, and Z_{ij} equal to the cofactor, in Z, of the element $r_{ij}(12\cdots k)$.

Comparison of the two expressions for U yields at once an argument that the array parameters $\sigma_j(12\cdots k)$, $r_{ij}(12\cdots k)$, $i, j=k+1, k+2, \cdots, n$, are *constants*, so that $\sigma_j(12\cdots k)=\sigma_{j,12\cdots k}$ and $r_{ij}(12\cdots k)=r_{ij,12\cdots k}$.

Moreover, since the regression of x_j on $(x_1, x_2, \cdots, x_k)$ has been shown to be linear, $j=k+1, k+2, \cdots, n$, we see that the parameter $\sigma_{j,12\cdots k}$ is identical with $\bar{\sigma}_{j,12\cdots k}$.

We conclude this article with an observation concerning normal distributions. Let C denote a constant, and let a distribution of the set $(x_1, x_2, \cdots, x_n)$ of n attributes, with all of n-space as domain, have the probability function Ce^{-T}, where T is any polynomial of second degree in $x_1, x_2, \cdots, x_n$ such that the terms of second degree constitute a positive-definite quadratic form in these n variables. Then the distribution is *normal;* and the constant C, as well as the parameters $\bar{x}_j$, σ_j, r_{ij}, $i, j=1, 2, \cdots, n$, may be determined without any integration whatever. The reader may easily outline the appropriate procedure.

SMALL OSCILLATIONS OF THE NEUTRAL HELIUM ATOM NEAR THE STRAIGHT LINE POSITIONS

By H. E. BUCHANAN, Tulane University

Introduction. The system considered in this paper consists of a central nucleus of mass m_2, carrying a positive charge of e_2, and two electrons of mass m_1 and m_3 carrying negative charges of $-e_1$ and $-e_3$ respectively. It is called the helium atom for the lack of a better name, the attention being centered solely on the mathematical discussion of the system. The gravitational attraction between the masses is neglected. We consider the special case in which $m_1=m_3$ and $e_2=2e=2e_1=2e_3$.

The author has previously discussed small oscillations of such a system near the equilateral triangle positions[1] and found that those positions are unstable. The mechanics of the problem of this earlier paper would lead one to expect that the straight line positions of the helium atom would be stable, but such is not the case. The instability is of the same nature as that of the straight line

[1] The AMERICAN MATHEMATICAL MONTHLY, vol. 39 (1931), p. 511.

positions of the three body problem discussed by the author[1] in the American Journal in 1923.

There are a number of interesting problems that occur in connection with this paper. The author takes the liberty of stating some of them.

1. The extension of this paper to the investigation of periodic orbits of the helium atom near the straight line positions when the masses are finite.

2. The discussion of the helium atom system when the gravitational attraction between the masses is considered.

3. In the three body problem for both the straight line case and the equilateral triangle case, and also in the helium atom problem for both cases, the characteristic exponents 0, 0, $\pm i\omega$, (ω being the angular velocity of the rotating axes) occur. Is this accidental? Are there certain types of U functions for which one could predict the occurrence of these roots of the characteristic equation?

The Circular Solutions. That there are circular solutions in which the nucleus and electrons remain in a straight line and revolve about their common center of gravity with angular velocity ω has been shown by Mr. W. E. Cox.[2] The angular velocity is $\omega=(e/2)\sqrt{7/m}$, where m is the mass of one electron and $2e$ is the charge on the nucleus. We assume that the units have been so chosen that the distance from each electron to the nucleus is unity and the proportionality factor in Coulomb's law, $k^2=1$.

The Differential Equations. If ξ_i, η_i and ζ_i, $i=1, 2, 3$, are the rectangular coordinates of the body m_i referred to axes rotating uniformly in the $\xi\eta$ plane with angular velocity ω, then the differential equations of motion are

$$
\begin{aligned}
&\frac{d^2\xi_i}{dt^2} - 2\omega\frac{d\eta_i}{dt} = \omega^2\xi_i + \frac{1}{m_i}\frac{\partial U}{\partial \xi_i},\\
&\frac{d^2\eta_i}{dt^2} + 2\omega\frac{d\xi_i}{dt} = \omega^2\eta_i + \frac{1}{m_i}\frac{\partial U}{\partial \eta_i},\\
&\frac{d^2\zeta_i}{dt^2} = \frac{1}{m_i}\frac{\partial U}{\partial \zeta_i}, \quad i = 1, 2, 3,\\
&U = \frac{e_1e_2}{r_{12}} - \frac{e_1e_3}{r_{13}} + \frac{e_2e_3}{r_{23}}.
\end{aligned}
\tag{1}
$$

Let $m_1=m_3=m$ and $2e=e_2=2e_1=2e_3$, and let the coordinates in the circular solutions be

$$\xi_1^{(0)} = -1, \quad \xi_2^{(0)} = 0, \quad \xi_3^{(0)} = 1, \quad \eta_i = \zeta_i = 0, \quad i = 1, 2, 3.$$

We make the transformation

$$\xi_1 = -1 + x_1, \quad \xi_2 = x_2, \quad \xi_3 = 1 + x_3, \quad \eta_i = y_i, \quad \zeta_i = z_i, \quad i = 1, 2, 3.$$

Wherever x_2, y_2 and z_2 occur they are eliminated by the center of gravity relations $m_1x_1+m_2x_2+m_3x_3=0$, $m_1y_1+m_2y_2+m_3y_3=0$ and $m_1z_1+m_2z_2+m_3z_3=0$. We

[1] American Journal of Mathematics, vol. 45 (1923), p. 93.

[2] The American Mathematical Monthly, vol. 40 (1933), p. 406.

expand the right members of the differential equations into a power series in x_1, x_3, y_1, y_3, and z_1, z_3. When only the linear terms are considered we have[1]

$$(2)\qquad \begin{cases} \dfrac{d^2x_1}{dt^2} - 2\omega\dfrac{dy_1}{dt} = A_{11}x_1 + A_{13}x_3 + B_{11}y_1 + B_{13}y_3, \\ \dfrac{d^2x_3}{dt^2} - 2\omega\dfrac{dy_3}{dt} = A_{31}x_1 + A_{33}x_3 + B_{31}y_1 + B_{33}y_3, \\ \dfrac{d^2y_1}{dt^2} + 2\omega\dfrac{dx_1}{dt} = C_{11}x_1 + C_{13}x_3 + D_{11}y_1 + D_{13}y_3, \\ \dfrac{d^2y_3}{dt^2} + 2\omega\dfrac{dx_3}{dt} = C_{31}x_1 + C_{33}x_3 + D_{31}y_1 + D_{32}y_3 \end{cases}$$

$$(3)\qquad \begin{cases} \dfrac{d^2z_1}{dt^2} = E_{11}z_1 + E_{13}z_3, \\ \dfrac{d^2z_3}{dt^2} = E_{31}z_1 + E_{33}z_3, \end{cases}$$

where

$$A_{11} = \frac{e^2(8m + 11m_2)}{2m\,m_2}, \qquad A_{13} = \frac{e^2(16m + m_2)}{4m\,m_2}$$

$$A_{31} = \frac{e^2(16m + m_2)}{4m\,m_2}, \qquad A_{33} = \frac{e^2(8m + 11m_2)}{2m\,m_2},$$

$$B_{11} = B_{13} = B_{21} = B_{33} = C_{11} = C_{13} = C_{21} = C_{33} = 0,$$

$$D_{11} = D_{13} = D_{31} = D_{33} = E_{13} = E_{31} = -\frac{e^2(16m + m_2)}{8m\,m_2},$$

$$E_{11} = E_{33} = -\frac{e^2(16m + 15m_2)}{8m\,m_2}.$$

The Characteristic Equation for the System (2). By the usual process the characteristic equation is found to be

$$(4)\qquad \begin{vmatrix} -\lambda^2 + \dfrac{e^2(8m + 11m_2)}{2m\,m_2}, & \dfrac{e^2(16m + m_2)}{4m\,m_2}, & 2\omega\lambda, & 0 \\ \dfrac{e^2(16m + m_2)}{4m\,m_2}, & -\lambda^2 + \dfrac{e^2(8m + 11m_2)}{2m\,m_2}, & 0, & 2\omega\lambda \\ -2\omega\lambda, & 0, & -\lambda^2 - \dfrac{e^2(16m + m_2)}{8m\,m_2}, & -\dfrac{e^2(16m + m_2)}{8m\,m_2} \\ 0, & -2\omega\lambda, & -\dfrac{e^2(16m + m_2)}{8m\,m_2}, & -\lambda^2 - \dfrac{e^2(16m + m_2)}{8mm_2} \end{vmatrix} = 0.$$

[1] The computations of these constants and the remainder of this paper have been checked by Miss Elizabeth Estorge as her master's dissertation.

Let $\lambda^2=(e^2x^2)/(mm_2)$ and substitute $\omega=(e/2)\sqrt{7/m}$, then (4) becomes, after multiplying the last two columns by $\sqrt{7m_2}$ and removing the common factors from the last two rows,

$$(5)\qquad \begin{vmatrix} -x^2+\frac{8m+11m_2}{2}, & \frac{16m+m_2}{4}, & 7m_2x, & 0 \\ \frac{16m+m_2}{4}, & -x^2+\frac{8m+11m_2}{2}, & 0, & 7m_2x \\ -x, & 0, & -x^2-\frac{16m+m_2}{8}, & -\frac{16m+m_2}{8} \\ 0, & -x, & -\frac{16m+m_2}{8}, & -x^2-\frac{16m+m_2}{8} \end{vmatrix} = 0.$$

In (5) subtraction of the last column from the third shows a common factor x and subtraction of the last row from the third shows another factor x. Hence $x=0$ is a double root. Removing the factor x^2 gives

$$\begin{vmatrix} -x^2+\frac{8m+11m_2}{2}, & \frac{16m+m_2}{4}, & 7m_2, & 0 \\ \frac{16m+m_2}{4}, & -x^2+\frac{8m+11m_2}{2}, & -7m_2, & 7m_2x \\ -1, & 1, & -2, & x \\ 0, & -x, & x, & -x^2-\frac{16m+m_2}{8} \end{vmatrix} = 0.$$

All the elements of the third row except the first may be made zero by proper manipulations and the following third order determinant is found:

$$\begin{vmatrix} -x^2+\frac{32m+23m_2}{4}, & 2x^2-8m-4m_2, & -x^3+\frac{8m+11m_2}{2}x \\ -x^2+\frac{32m+23m_2}{4}, & -\frac{16m+15m_2}{2}, & \frac{16m+29m_2}{4}x \\ -x, & x, & -x^2-\frac{16m+m_2}{8} \end{vmatrix} = 0.$$

Adding the second column to the first shows the common factor $x^2+7m_2/4$. Hence another pair of roots of (4) is $\lambda=\pm\iota\omega$ where $\iota=\sqrt{-1}$.

Expanding and arranging according to powers of x gives

$$x^4+\left(-4m+\frac{3m_2}{2}\right)x^2-\frac{512m^2+400mm_2+23m_2^2}{16}=0.$$

Hence

$$x^2=\frac{8m-3m_2}{2}\pm 2\sqrt{36m^2+22mm_2+2m_2^2}.$$

Since m is very small compared to m_2, x^2 certainly has one negative and one positive value. Let these values be represented by $-\nu^2 mm_2/e^2$ and $\rho^2 mm_2/e^2$. Then (4) has the roots

$$0, 0, \pm \iota\omega, \pm \iota\nu, \pm \rho.$$

The Characteristic Equation for the System (3). The usual method gives the following characteristic equation for the z_i equations

$$(6) \qquad \begin{vmatrix} -\lambda^2 - \dfrac{e^2(16m + 15m_2)}{8mm_2}, & -\dfrac{e^2(16m + m_2)}{8mm_2} \\ -\dfrac{e^2(16m + m_2)}{8mm_2}, & -\lambda^2 - \dfrac{e^2(16m + 15m_2)}{8mm_2} \end{vmatrix} = 0.$$

The solution of this for λ^2 gives

$$\lambda^2 = -\frac{7e^2}{4m} = -\omega^2 \text{ and } \lambda^2 = -\frac{2e^2(2m + m_2)}{mm_2} = -\sigma^2.$$

Hence the roots of (6) are $\pm\iota\omega$, $\pm\iota\sigma$.

It is interesting that the value of σ is the same as the angular velocity in the equilateral triangle case.[1]

The Solutions of Equations (2) *and* (3). From the values of the characteristic exponents found above, we may write the solutions of equations (2) and (3)

$$(7) \qquad \begin{aligned} x_i &= K_{i1} + K_{i2}t + K_{i3}e^{\omega\iota t} + K_{i4}e^{-\omega\iota t} + K_{i5}e^{\nu\iota t} + K_{i6}e^{-\nu\iota t} + K_{i7}e^{\rho\iota t} + K_{i8}e^{-\rho\iota t}, \\ y_i &= L_{i1} + L_{i2}t + L_{i3}e^{\omega\iota t} + L_{i4}e^{-\omega\iota t} + L_{i5}e^{\nu\iota t} + L_{i6}e^{-\nu\iota t} + L_{i7}e^{\rho\iota t} + L_{i8}e^{-\rho\iota t}, \end{aligned}$$

$$(8) \qquad z_i = M_{i1}e^{\omega\iota t} + M_{i2}e^{-\omega\iota t} + M_{i3}e^{\sigma\iota t} + M_{i4}e^{-\sigma\iota t}, \quad i = 1, 3,$$

and their derivatives.

We may choose six of the K_{ij}, L_{ij}, $i=1, 3$, $j=3 \cdots 8$ arbitrarily and the remainder are determined uniquely by well known methods. Similarly we may choose m_{ij}, $j=1 \cdots 4$ arbitrarily and M_{3j} are uniquely determined. The determination of the K_{ij} and L_{ij}, $i=1, 3$, $j=1, 2$ is of interest.

Substitute

$$x_i = K_{i1} + K_{i2}t, \quad y_i = L_{i1} + L_{i2}t, \quad i = 1, 3$$

in equations (2). There results:

$$\begin{aligned} -2\omega L_{12} &= A_{11}(K_{11} + K_{12}t) + A_{13}(K_{31} + K_{32}t), \\ -2\omega L_{32} &= A_{31}(K_{11} + K_{12}t) + A_{33}(K_{31} + K_{32}t), \\ 2\omega K_{12} &= D_{11}(L_{11} + L_{12}t) + D_{13}(L_{31} + L_{32}t). \\ 2\omega K_{32} &= D_{31}(L_{11} + L_{12}t) + D_{33}(L_{31} + L_{32}t). \end{aligned}$$

[1] Buchanan, AMERICAN MATHEMATICAL MONTHLY, vol. 39 (1931), p. 511.

These equations must be identities in t, hence

$$(9)\quad \begin{cases} -2\omega L_{12} = A_{11}K_{11} + A_{13}K_{31}, \\ -2\omega L_{32} = A_{31}K_{11} + A_{33}K_{31}, \end{cases} \qquad (10)\quad \begin{cases} A_{11}K_{12} + A_{13}K_{32} = 0, \\ A_{31}K_{12} + A_{33}K_{32} = 0, \end{cases}$$

$$(12)\quad \begin{cases} 2\omega K_{12} = D_{11}L_{11} + D_{13}L_{31}, \\ 2\omega K_{32} = D_{31}L_{11} + D_{33}L_{31}, \end{cases} \qquad (13)\quad \begin{cases} D_{11}L_{12} + D_{13}L_{32} = 0, \\ D_{31}L_{12} + D_{33}L_{32} = 0. \end{cases}$$

Since the determinant of K_{12} and K_{32} in (10) does not vanish, $K_{12}=K_{32}=0$. For the same reason (9) determines K_{11} and K_{31} uniquely in terms of L_{12} and L_{32}. Since the determinant of L_{12} and L_{32} in (12) and (13) does vanish L_{32} is uniquely determined in terms of L_{12} and L_{31} uniquely in terms of L_{11}. Therefore, it is better to choose L_{ij}, $j=1 \cdots 8$ as the arbitrary constants and all the others are uniquely determined in terms of these.

Types of Periodic Oscillations. An examination of equations (7) and (8) shows the following types of oscillations:

(1) Oscillations with period $2\pi/\omega$.
(2) Oscillations with period $2\pi/\nu$.
(3) Oscillations with period $2\pi/\sigma$.
(4) Oscillations which are combinations of any two or three of these.

This paper does not show whether these oscillations persist when all the terms in the expansions of the right members of equations (1) are taken into account. It is probable that there are periodic orbits with each of these periods for properly chosen initial conditions.

If we take $m_2=1$ and $m=1/1800$ then we have approximately

$$\omega = 56.1e, \quad \nu = e/20.9, \quad \sigma = 2e.$$

Thus by far the shortest period of oscillation is $2\pi/\omega$. The three periods are approximately proportional to the numbers, .112, 131, and 3.14.

A NEW METHOD FOR FINDING THE NUMERICAL SUM OF AN INFINITE SERIES

By C. C. CAMP, University of Nebraska

1. *Introduction.* Although the remainder R_n after taking n terms of an alternating series is numerically less than the next term it is not always so easy to get any desired accuracy. The Euler transformation is of limited application and is rather complicated. In the case of positive series the methods of Kummer and Markoff are even more laborious. The Maclaurin-Cauchy Integral Theorem gives what might be called a first approximation to upper and lower bounds for R_n in such series. The aim of the present paper is to present closer bounds by a simple formula and to give a very easy rule for interpolating so that most series can be evaluated without too much arithmetic to five or six decimals. Accuracy

to ten decimals may be secured by taking into account a certain asymptotic error.

In §2 the formula is derived from a geometrical point of view. Various types of extrapolation and interpolation are considered in §3 together with the so-called rule and its error. The other three sections treat alternating series, asymptotic forms and infinite products.

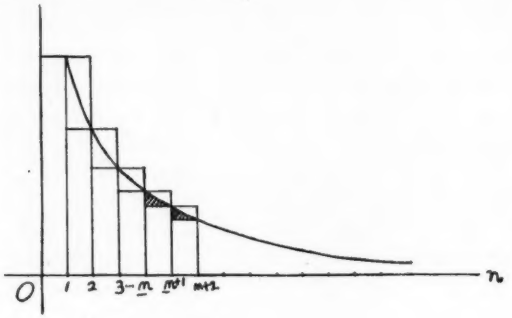

Figure 1.

2. *Derivation of the Fundamental Inequalities.* Define u_n for non-integral values of n so as to get a continuous function which tends steadily to zero. For n an integer the ordinate is the corresponding term of the series $\sum_{n=1}^{\infty} u_n$. Using exterior rectangles one sees readily that

$$\int_{n+1}^{\infty} u_n dn + \tfrac{1}{2}u_{n+1} < R_n$$

provided the curve lies below its chords. This restriction on the curve (and the series) is sufficient to insure the inequality, but not necessary. Using interior rectangles and defining k_n as the ratio of the total of the shaded areas[1] in figure 1 to the sum of the two rectangles containing them one sees that

$$R_n + k_n u_n < \int_{n}^{\infty} u_n dn,$$

[1] The use of two shaded areas rather than one is made for two reasons. First, it makes the approximate determination by (4) in section 4 much simpler for the case in which u_n is difficult to integrate. Secondly, because it leads to much greater accuracy after interpolation by the rule in section 3, than other definitions involving simple ratios of areas. The ratio k'_{n+1} of the second shaded area to its containing rectangle gives a smaller difference between the upper and lower bounds for R_{n+1}, which would be an advantage when p is unknown. However, for Σn^{-p} the error after interpolation is asymptotic to $p/\{24(n+2)^{p+2}\} + 2(2p-1)(5-p)/\{240(n+2)^{p+3}\}$, instead of $p(3-p)/\{60(n+1)^{p+2}\}$, which for $p=2$, $n=10$, is over twenty times as great, namely 0.0000043.

The error resulting from (4) section 4 is close to $p(p+2)(p+3)/\{180(n+1)^3 n^p\}$. The algebra required to derive the asymptotic results in this paper is rather tedious and is omitted to save space.

provided we now restrict our series so that k_n will increase with n. This is the usual case for slowly converging series, and in particular for $u_n = c\,n^{-p}$, $p>1$, as well as for series whose general term is asymptotic to such an expression. The two inequalities may be combined in the formula

$$\int_{n+1}^{\infty} u_n dn + \tfrac{1}{2}u_{n+1} < R_n < \int_{n}^{\infty} u_n dn - k_n u_n, \tag{1}$$

where

$$k_n = \left(\int_{n}^{n+2} u_n dn - u_{n+1} - u_{n+2}\right)/(u_n - u_{n+2}). \tag{2}$$

3. It can be shown that the upper bound a_1 for the series mentioned above is nearer the true value than the lower b_1 for the sum s of the series. Consequently if one calls the difference Δ_1 the value $a_1-\Delta_1/4$ is in error less than $\Delta_1/4$ numerically. By experiment it was found that $a_1-\Delta_1/3$ was a still closer value for the case $\sum_{n=1}^{\infty} n^{-2}=\pi^2/6$. Indeed for $n=10$ this gives 1.6449343 with an error of 2×10^{-7} and for $n=20$ the interpolated value was correct to 8 decimals when terms were carried to nine.

One may use (1) for two different values of n, add s_n, then extrapolate by using straight lines to get a closer value for s, namely

$$t = b_2 + (b_2 - b_1)\Delta_2/(\Delta_2 - \Delta_1), \tag{3}$$

where the subscript 2 refers to the larger value of n. For $n_1=5$, $n_2=10$ this gives $t=1.6449335$ with an error of -6×10^{-7}. This is less accurate than the simple rule for interpolation, which may be stated thus:

Rule for Interpolation: For $\sum_{n=1}^{\infty} n^{-p}$ calculate a_1, b_1 from (1), then take $a_1-\Delta_1/(p+1)$. This may be verified by using the Euler-Maclaurin sum formula. Moreover the error made in using this simple rule is asymptotic to $p(3-p)/\{60(n+1)^{p+3}\}$. The maximum numerical error for any value of p and $n=10$ is 2.3×10^{-6}. For $p=2$, $n=10$ it is 2.070×10^{-7}. By using nine decimal places for $p=3$ and taking k_{10} to seven one gets readily $\sum_{n=1}^{\infty} n^{-3}=1.202056904$ with an accidental error of 1×10^{-9}. For $p=4$ the error in the rule is -3.42×10^{-9}. By taking this into account one may carry the terms to 12 decimals and obtain $\sum n^{-4}=1.082323233655$ correct for $\pi^4/90$ when rounded to ten decimal places.

4. *Alternating Series.* Even slowly convergent alternating series may be evaluated easily by pairing terms. For instance $\pi/4=1-1/3+1/5-1/7+\cdots$ or $\pi/8=1/(1\cdot3)+1/(5\cdot7)+\cdots+1/\{(4n-3)(4n-1)\}+\cdots$. Calculation of t for $n=5$, 10 gave 0.392697 with an error of -2×10^{-6}. The use of the rule for the dominant part of this, namely $\sum 1/(16n^2)$ gave the same error. Similarly the series $\log 2=1-1/2+1/3-1/4+\cdots=\sum 1/\{2n(2n-1)\}$ gave $t=0.6931485$ with an error of 1.3×10^{-6} whereas $a_2-\Delta_2/3$ gave six decimals correct. It is interesting to note that for these series k_5 and k_{10} are practically the same as for the case $p=2$ above.

5. *Asymptotic Forms.* Series whose general term contains a factorial may be handled by substituting the gamma function and using the asymptotic form

$$\log \Gamma(n) = (n - \tfrac{1}{2}) \log n + \tfrac{1}{2} \log 2\pi - n + 1/(12n) - 1/(360n^3) + 1/(1260n^5) - \cdots$$

to simplify u_n in (1) before integration. When $n \geqq 10$ the results are accurate enough for practical use. To calculate (2) one may use Simpson's Rule and write as an approximation

$$k_n = (u_n + u_{n+1} - 2u_{n+2})/3(u_n - u_{n+2}). \tag{4}$$

If greater accuracy is required we may take $n > 10$ and use (2) and if necessary more terms of Stirling's Series. The series

$$\Sigma \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!(8n+1)}$$

arises in the term by term integration of

$$\int_0^1 (1 - r^8)^{1/2} dr = \int_0^{\pi/2} (\sin x)^{5/4} dx,$$

where $\sin x = r^4$. The value of the integral is $1 - s$, where s is the sum of the series. The general term may be written

$$u_n = \left(n^{-5/2} + \frac{n^{-7/2}}{4} + \frac{21n^{-9/2}}{128} + \frac{43n^{-11/2}}{512} + \cdots \right) / 16\pi^{1/2}.$$

If the terms are carried to nine decimals and we take $k_8 = .45118$, $k_9 = .47067$, by Simpson's Rule then $t = .0691260$. If we take $p = 5/2$ for the dominant term and use the rule for interpolation we obtain 0.0691259 all of whose digits are correct. The true value is $s = 0.069125943$ and the original integral has the value $1 - s = \Gamma(1.5)\Gamma(1.125)/\Gamma(1.625)$, which was evaluated with Legendre's 14-figure tables. It is very clear that the rule for interpolation entails much less arithmetic. The other is useful, however, as a control.

As a crucial test consider Maclaurin's series for $\sin^{-1} x$ and put $x = 1$ obtaining

$$\sin^{-1} 1 = 1 + 1/(2 \cdot 3) + 1 \cdot 3/(2 \cdot 4 \cdot 5) + \cdots + 1 \cdot 3 \cdots (2n-3)/\{2 \cdot 4 \cdots (2n-2)(2n-1)\} + \cdots.$$

This has been characterized by Bromwich as converging "so slowly as to be quite unsuitable for numerical computation." Here

$$u_n = \Gamma(2n-1)/\{2^{2n-2}[\Gamma(n)]^2(2n-1)\}.$$

As before we get the asymptotic form

$$u_n = (n^{-3/2} + 7n^{-5/2}/8 + 81n^{-7/2}/128 + 429n^{-9/2}/1024 + \cdots)/2\pi^{1/2}.$$

$$\int_0^\infty u_n dn = (1 + 7/24n + 81/640n^2 + 429/7168n^3 + \cdots)/(\pi n)^{1/2}.$$

The terms may be exhibited as follows:

1.370	381	235	$= s_8$
1.000	000	000	
.166	666	667	
.075			
.044	642	857	
.030	381	944	
.022	372	159	
.017	352	764	
.013	964	844	$= u_8$
.011	551	801	$= u_9$
.009	761	610	
.008	390	336	
.007	312	526	
1.407	397	508	$= s_{12}$
.006	447	210	$= u_{13}$
.005	740	038	$= u_{14}$

By the approximate formula (4) one has

$$k_8 = 0.4753027 \quad k_{12} = 0.483238.$$
$$b_1 = 1.5706242 < s < 1.5709050 = a_1$$
$$b_2 = 1.5707313 < s < 1.5708385 = a_2$$
$$t = 1.5707974 \text{ with an error of } 1.1 \times 10^{-6}.$$

For $p = 3/2$ the rule gives a numerical error of about 7×10^{-7}, less than 1/9,000 of the next term u_{13}. Greater accuracy would probably be obtained with the true value of k_{12} as given in (2).

6. *Infinite Products.* By taking the logarithm of an infinite product one obtains an infinite series which may be summed by the method of this paper. For example

$$P = \prod_{n=1}^{\infty} (1 + 1/n^2) = \sinh \pi/\pi$$

is treated by using $\log P = \sum \log (1 + n^{-2})$.

$$u_n = \log (n^2 + 1) - 2 \log n.$$

$$\int u_n dn = n \log (1 + n^{-2}) + 2 \tan^{-1} n.$$

Seven-place common logarithms were used to sum the first ten terms. $k_{10}=0.477164$. Strictly $u_n=n^{-2}-n^{-4}/2+\cdots$ could be summed more accurately by taking several series with different values for p. For practical purposes, however, one takes $p=2$. The rule then gives 3.676081 as against the true value 3.676078. The sum of the series for log P was in error approximately 8×10^{-7}. It is clear that finite products for n large as well as finite series with a great number of terms can quite readily be evaluated by adaptations of the methods given above.

THE NUMBER SYSTEM OF THE MAYAS[1]

By A. W. RICHESON, University of Maryland

The Number Systems of the North American Indians have recently been discussed in detail in two papers in this monthly.[2] The system of numbers developed by the semi-civilized Maya Indians of Central America is probably the most interesting of all systems developed by the early inhabitants of this continent.

The examples of the number system of the Mayas that have been found, or at least that have been deciphered, deal with the counting of time events or periods, and many authorities are of the opinion that the recording of time series was the sole purpose of their numbers. The records of their chronicles are found as glyphs on the monuments and as written in the codices. These records present two methods of writing numerals, the normal form and the head-variant form. Both forms are essentially the same, and the Mayas were able to express a number as easily by one method as by the other. The head-variant form is found with few exceptions on the monuments, while the normal form is found exclusively in the codices.

In the head-variant form there are distinctive head forms for each of the numbers from 0 to 12 inclusive, while from 13 to 19 inclusive the numbers are written by using the head form for 10 plus the form for whatever unit is needed to make up the desired number. Each number is characterized by a distinctive type of head, by means of which it can be distinguished from any other number. In the case of three numbers, 2, 11, and 12, however, the characteristic elements have not been determined with certainty. The forms for these numerals occur

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[1] It is impossible to give complete references for many of the statements, but the material has been drawn largely from the following sources, except where specific references are given: S. G. Morley, *An Introduction to the Study of Maya Hieroglyphs*, Bureau of American Ethnology, Bulletin 57; Cyrus Thomas, *Numerical Systems of Central America and Mexico*, 19th Annual Report of the Bureau of American Ethnology, 1897–98, pp. 853–955; Spinden, *Ancient Civilizations of Mexico*; W. J. McGee, *Primitive Numbers*, 19th Annual Report of the Bureau of American Ethnology, pp. 821–851; S. G. Morley, *The Inscriptions at Copan*, Carnegie Institution, Washington, D. C., 1920; John Teeple, *Maya Astronomy*, No. II, Carnegie Institution, Washington, D. C.

[2] Eels, *Number Systems of the N. A. Indians*. This MONTHLY, vol. 20 (1913), pp. 263–279; pp. 292–299.

very rarely on the inscriptions, and consequently, the data are not sufficient to justify a statement as to the characteristic elements.

Figures 1–3 illustrate the head forms for 6, 10, and 16 respectively. Figure 4 shows the head form for 16 used as a multiplier with the kin or day sign to the

Fig 1

Fig 2

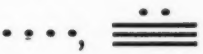
Fig 3

Fig 4

right. It should be noted that the head form for 16 is made up of the fleshless jaws of the character for 10 with the "hatchet" eye for 6. The characteristic elements for the numbers from 0 to 19 are given below:

Head form for	*Characteristic element*
0	Clasped hands across lower part of face.
1	Forehead ornament composed of more than one part.
2	Undetermined.
3	Banded head dress.
4	Bulging eye with square irid, snag tooth, curling fangs from back of mouth.
5	"Tun" sign for head dress.
6	Hatchet eye.
7	Large scroll passing under eye and curling under forehead.
8	Forehead ornament composed of one part.
9	Dots on lower cheek or around mouth.
10	Fleshless lower and upper jaws.
11	Undetermined.
12	Undetermined—type of head known.
13 to 19	Head for 3, 4, 5, 6, 7, 8, 9 with fleshless lower jaw for 10.

In the normal form the number combinations from 1 to 19 inclusive are formed by dots and bars. Each dot has the numerical value of 1 and each bar represents five. Generally the dots are placed horizontally over the bars or to the side of a vertical arrangement of the bars; for example, 4 and 17 were written respectively as follows:



On the inscriptions the number forms were frequently decorated to give them symmetry and a balanced form; this has often been a source of error in deciphering the inscriptions. Since the Mayas used a vigesimal system of numeration, there was no need of a symbol for twenty, since 20 units of the first order

gave one unit of the second. However, a symbol for zero was absolutely indispensable, and this symbol, which somewhat resembled the shape of a shell, is found on the inscriptions and in the codices. The symbol was first recognized by Dr. Förstemann.[1]

Methods of numeration: The Mayas developed two systems of numeration; the multiplication method and the "numeration by position." Although different in form, both methods are essentially vigesimal.

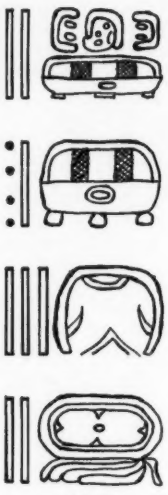


Fig 5

The first method, which is rarely found except on the inscriptions, makes use of both the normal and head-variant forms. The numbers are formed by using the bar and dot characters or the desired head form to build up the multipliers from 0 to 19 inclusive, with the time period signs as multiplicands. Until recently most authorities have stated that the Maya time count was one of days or kins and that the count was not strictly vigesimal. The following table will show the count under this assumption:

	1 kin	=	1 day
20 kins	= 1 uinal	=	20 days
18 uinals	= 1 tun	=	360 days
20 tuns	= 1 katun	=	7,200 days
20 katuns	= 1 baktun	=	144,000 days
etc.			

On the other hand, Dr. Teeple of the Carnegie Institute and Mr. Wm. E. Gates of The Johns Hopkins University have advanced the opinion that the tun is the correct unit of time used by the Mayas, and that the time count is vigesimal

[1] Förstemann, *Zur Maya Chronologie*, Zeitschrift für Ethnologie, (Berlin) 1891.

throughout. They argue that the division of the tun or year into 18 and 20 parts is nothing more than fractional parts of the Maya time unit.[1]

Figure 5 illustrates the formation of the number 75,500 on the basis of the above table by employing the dot and bar characters for the multipliers with the kin, uinal, tun, and katun signs as multiplicands. Reading from the top down, we have 10 katuns = 72,000 kins, 9 tuns = 3,240 kins, 15 uinals = 300 kins, and 10 kins = 10 kins. The sum of the four products is 75,500 kins. This number could be expressed also by the head forms for the multipliers 10, 9, 15, and 10, in place of the dot and bar characters.

The second method of numeration, namely that by position, is very similar to the Hindu-Arabic decimal system as used today. Although the system is vigesimal and thus required 19 different combinations for the units, it was built up by the three simple characters: the dot, the bar, and the zero. With this method the Mayas were forced to fix arbitrarily a starting point and to confine themselves to one series only, or else the positional value of the nineteen digits would be useless. They accordingly adopted an ascending series which corresponds to our decimal series from right to left.

Let us illustrate the method by the number 12,489,781

• • • •	= 4×	2,880,000 =	11,520,000
—•	= 6×	144,000 =	864,000
═ • • • •	= 14×	7,200 =	100,800
═ • • •	= 13×	360 =	4,680
═══	= 15×	20 =	300
•	= 1×	1 =	1
			12,489,781

This is the largest number yet found in the codices.

Discussion of the numbers: The Maya numbers were no doubt written as they were spoken. The names of the numbers from 1 to 20 inclusive are given below as they appear in Beltran's *Arte del Idioma Maya*.

1 hun	11 buluc
2 ca	12 lahca
3 ox	13 oxlahun
4 can	14 canlahun
5 ho	15 lolahun
6 uac	16 uaclahun
7 uuc	17 uuclahun
8 uaxac	18 uaxaclahun
9 bolon	19 bolonlahun
10 lahun	20 hunkal or kal

Very little seems to be known concerning the origin of the number words from 1 to 5 inclusive. Dr. Brinton is of the opinion that the Maya proper and the neighboring Mayan dialects were derived from one common archaic form

[1] Teeple, *Maya Astronomy*, Carnegie Institution of Washington, D. C.

of speech and not from one another. In the case of the smaller numbers, this opinion seems to be justified, as they were no doubt formed before the beginning of their history. A number of arguments have been advanced for the derivation of the numbers from 5 to 9. Dr. Thomas believes that the hand was not used in the count until 5 was reached and that the numbers from 6 to 8 inclusive were composite. He suggests that uaxac is the answer to the whole question, that the *x* of ox = 3 has been combined by *u* with some form of 5 to give eight and that the forms for 6 and 7 are formed in a similar manner. Pio Perez on the other hand gives as a signification of the verb uac or uach "to take out one thing which is placed in another and united with it." This would seem to indicate counting on the fingers and turning them in for the first five and then opening them out while counting the second five. Bolon = 9 seems to have the meaning "on the way to 10," while lahun = 10 is lah hun; it finishes one man, i.e., counting on the fingers.

The numbers from 12 to 19 inclusive are without doubt composite numbers, i.e., 12 = 10 + 2, 13 = 10 + 3, etc. As we should expect in a vigesimal system, there is a definite number for 20, kal or hunkal. Henderson gives for kal "to close, to shut" or as a substantive "a fastening together," i.e., a fastening together of both hands and feet.

The count from 21 to 40 inclusive is by addition to the first 20, e.g., 21 is hun-tu-kal = 1 + 20 or 1 to the 20. Forty is ca-kal or 2 × 20. From 41 on, the count is regular, but is different from 21 to 40. Here the count is by subtraction from the next 20.[1]

Dr. Brinton states that the Maya's use of numbers was somewhat different from ours.[2] The numbers are rarely used except with a numeral particle, which is suffixed to the numeral and indicates the character or class of the objects which are about to be enumerated. With the aid of these particles Dr. Brinton gives another method which was frequently employed to express their numbers. For eighty-one years they did not write hun tu yox kal haab, as we would expect, but can kal haab catac hunpel haab, i.e., four score years and one year.

Conclusion: It is quite evident that the Mayas had developed a number system with a place value for their characters many years before the advent of the white man. Dr. Teeple is of the opinion that the Mayan vigesimal system of numbers was a distinct part of an American civilization.[3]

The records also indicate that the Mayas were unable to handle fractions as we do today, but on the other hand they were able to and did perform long numerical computations involving multiplication and division. Just how these computations were carried out we do not know. Dr. Förstemann in his works gives an instance from the inscriptions where the calculation runs into the millions.[4]

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[1] Rosney, *Mémoire sur la numération dans la langue et dans l'écriture sacrée des anciens Mayas*, Compte-Rendu de Congrès International des Américanistes, (Paris 1875), vol. 2, p. 439.

[2] Brinton, *Maya Chronicles*, pp. 49–50.

[3] Teeple, Ibid, p. 31.

[4] Förstemann, *Zur Entsifferung der Maya-Handschriften*, No. II.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

A SIMPLE CONTINUOUS FUNCTION WITH A FINITE DERIVATIVE AT NO POINT

BY T. H HILDEBRANDT, University of Michigan

Van der Waerden[1] has given a simple example of a continuous function which at no point has a finite derivative. A slight modification of his example produces a still simpler instance.

Define $f_0(x)$ of period 1, so that

$$f_0(x) = x \text{ for } 0 \leqq x \leqq \tfrac{1}{2}$$
$$= 1 - x \text{ for } \tfrac{1}{2} \leqq x \leqq 1,$$

$f_0(x)$ forms with the x-axis a set of right-angled isosceles triangles with vertices at $(\pm m, 0)$ and $(\pm m+\frac{1}{2}, \frac{1}{2})$ $m=0, 1, 2, \cdots$. Let $f_n(x)=f_0(2^n x)/2^n$. Then obviously $\phi(x)=\sum_0^\infty f_n(x)$ is a uniformly convergent series of continuous functions and therefore continuous.

To show that $\phi(x)$ has at no point a finite derivative, we make use of the well known lemma: If $\phi'(x)$ exists then

$$\lim_{(h,k)\to(0,0)} \frac{\phi(x+h)-\phi(x-k)}{h+k} = \phi'(x), \quad h, k \geqq 0.$$

As a consequence if for every x_0 we can determine within every interval enclosing x_0 two pairs of points for which the corresponding secants on $\phi(x)$ differ by unity, $\varphi'(x_0)$ cannot exist as a finite number.

We note that $f_n(x)=0$ for $x=p/2^n$, i.e. the value of $\phi(p/2^n)$ is determined by the functions $f_1(x), \cdots, f_k(x)$, $k<n$. Further since the slope of $f_0(x)$ is constant on $(m, m+\frac{1}{2})$, and $(m+\frac{1}{2}, m+1)$ and consequently on any interval of the form $(p/2^m, (p+1)/2^m)$, for $m>0$, it follows that the same is true of $f_n(x)$ on any interval of the form $(p/2^{n+m}, (p+1)/2^{n+m})$ for $m>0$. Then the slope of the line joining $[(2p+1)/2^{n+1}, \phi\{(2p+1)/2^{n+1}\}]$ and $[p/2^n, \phi\{p/2^n\}]$ differs from that of the line joining $[p/2^n, \phi\{p/2^n\}]$ and $[(p+1)/2^n, \phi\{(p+1)/2^n\}]$ by the slope of $f_n(x)$ between $x=p/2^n$ and $x=(2p+1)/2^{n+1}$, viz. 1. A similar statement holds if $2p+1$ and $p+1$ are replaced by $2p-1$ and $p-1$ respectively. These facts are also immediately evident from the graphical representation of a few of the approximating functions of $\phi(x)$. Since for any x and n, there exists an integer p such that $p/2^n \leqq x < (p+1)/2^n$, the difference quotient $\{\phi(x+h)-\phi(x-k)\}/(h+k)$, $h, k \geqq 0$, cannot approach a finite limit as $(h, k)\to(0, 0)$, for any x, i.e. if $\phi'(x)$ exists, it is infinite.

[1] Mathematische Zeitschrift, vol. 22 (1930, pp. 474–5.

Note by the Editor. This function of Professor Hildebrandt seems to be an admirably simple example of a continuous function which at no point possesses a finite derivative. The following interesting and probably not too difficult question is suggested. For what values of x (if any) does $\phi(x)$ have a definite infinite derivative, that is to say, for what values of x does

$$\frac{\phi(x+h)-\phi(x-k)}{h+k} \to +\infty \text{ or } \to -\infty$$

as $h, k \to 0$, $h, k \geqq 0$?

Will some reader supply the answer?

RECENT PUBLICATIONS

EDITED BY R. A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Craftsmanship in the Teaching of Elementary Mathematics. By F. W. Westaway. London, Blackie and Son, 1931. xiv+665 pages. 15s net.

Mr. Westaway, a former British Inspector of Secondary Schools, sets down in this book the results of his long experience as friendly critic of young teachers of mathematics. The tone is authoritative, but not dogmatic. In many instances his judgments proceed from a balancing of one method against another in a way that appeals at once to the reader's common sense. On occasion his discussion of several alternative methods appears to be impartial when in reality it is mildy prejudiced. The reviewer usually shares Mr. Westaway's prejudice in such instances, but believes that mathematics and justice would have been better served if the author had taken more pains to exhibit the strong points of methods which he eventually rejects. A good example of this is his treatment of subtraction in arithmetic, which falls far short of Thorndike's treatment of the same topic in his Psychology of Arithmetic. On the other hand Mr. Westaway has covered a great deal of ground, interpreting Elementary Mathematics to embrace everything from the beginning of arithmetic through the elements of the calculus, including some mention of mechanics, astronomy, non-Euclidean geometry, and the philosophy of mathematics; and there is much helpful comment on all these topics. The reviewer's criticism above ought to be construed, therefore, merely as a description of the book and its scope, and not as chiding the author for giving us only six hundred pages of good counsel when he might have given us one or two hundred more. For those who wish to pursue a given matter further, there are references to other books.

The advice he gives will benefit most the relatively inexperienced teacher, and hence indirectly his earlier pupils also; but there is much to interest and challenge the teacher of experience also. Many of his comments might well be taken to heart by teachers and administrative officers in the United States. He scores the British practice of assigning teachers untrained in mathematics to classes in arithmetic and beginning algebra. The reviewer wonders if the British offend in this respect more than we do. Mr. Westaway admits that mathematics is not difficult to teach, but he wants teachers to be conversant with their subject and trained to teach it; furthermore he feels that a degree of experience is necessary, so that actually there are few good teachers of mathematics under thirty-five years of age. He regrets that there are almost no expert mathematicians in the training colleges for teachers; but he asserts that this need not be so, and that it is possible to make the training course of great value to the embryo teacher under the tutelage of men like Professor Sir Percy Nunn, to whom he acknowledges his indebtedness in many ways. He would assign the green teacher to the upper classes, insuring to the younger pupils competent instruction by teachers of experience. This has always seemed to the reviewer the wiser plan, regardless of the college entrance bugbear: older students, trained in the ways of the school, can protect themselves against the beginning teacher's earlier blunders and can be of real assistance in "breaking him in." Mr. Westaway suggests that the embryo teacher begin in a large strong school under competent guidance, rather than be thrust out into a small isolated school with responsibility from the very outset for much or all of the mathematics. This latter practice, so common with us, is fair neither to the teacher nor his pupils. In New Zealand these isolated posts are the plums given to the most experienced. Mr. Westaway is fearful lest democracy in education, like democracy in other fields, may become so interested in standardization as to stifle original methods, initiative, and intellectual independence.

The arrangement of the book is simple and obvious: a few brief chapters on teachers and methods at the start, followed by consideration of the various topics in mathematics and related fields from the kindergarten to the last years of the secondary school. Under method in mathematics he argues for that particular presentation which carries most meaning to the student, regardless of whether an expert mathematician would call it mathematically neat, or not. He would postpone all abstractions until the practical aspect of a subject had been made clear to the student. This sounds reasonable enough, and yet Mr. Westaway's exemplification of this innocent declaration seems to the reviewer to involve probably its only exception. For he asserts that he would defer all discussion of the postulates of demonstrative geometry until the very last year of school; and while his intent here is obvious and wise, he makes no provision for explaining to beginners in demonstrative geometry in some simple way that one of the advertised aims of instruction in this subject is to develop the powers of logical deduction, and that logical deductions necessarily start from certain assumptions. He has previously complimented the training colleges for

disabusing young teachers of the idea that their subject was the only one in the curriculum. Why not lead young pupils to appreciate also that their geometry is not the only geometry? It is not necessary to mention non-Euclidean geometry in this connection.

He pleads for teaching "honest thinking" through mathematics and laments the passing of Euclid. He would restore him to the schools, "frankly admitting his failings and promising reform." He admits that under the old system boys did not understand what they were doing and knew little of actual geometry; nevertheless he asserts that "they were learning to think logically and produce good authority for every assertion they made." This is reminiscent of Bertrand Russell's characterization of mathematics as the subject in which we are never sure what we are talking about, and in the same vein could lead so easily to an understanding of that very remark. It would seem as though the author ought to have taken the opportunity to discuss with the beginning teacher the whole question of demonstrative geometry in relation to "understanding" and "logical thinking," and have made some reference to the possibility of transferring this power of logical thought from situations in which it is developed to situations in which it will be applied.

The book suffers somewhat from a certain incoherence and vagueness which is a bit disconcerting. It gives the effect of being a sort of scrap book, full of important matters which are elucidated only in part. More than once one meets a detailed explanation which seems to tell only half the story, and wonders why Mr. Westaway holds back the rest. Later one finds a casual admission that he knows the sequel, and the matter is closed. He warns the reader at the outset that he will turn without notice from addressing pupils to addressing teachers. This sufficiently explains the style, without making it ideal for the young teacher. For the latter wants to know the whole story, and in detail; and is just as much interested in the scandal for adult ears only, which Mr. Westaway often glosses over, as in the expurgated edition for the innocent. But half a loaf is better than no bread, and nobody can claim to be starved who has turned to this source for help. The reviewer will indicate some of the topics where he wishes fuller treatment could have been given, and will let this suffice to indicate the general scope of the book.

In discussing mathematical reasoning (pp. 13, 14) the author says, "Mathematical reasoning is not, as commonly supposed, *deductive* reasoning; it is based upon an initial analysis of the given and, being analytical, is in essence *inductive*." And almost immediately thereafter, "In fact, the main source of fallacious reasoning almost always lies in false premises. The truth of the conclusion cannot be more true than the truth of the premises, and a scrutiny and a rigorous analysis of these is therefore always necessary. At bottom, all reasoning is much of the same kind, and it usually turns on the truth or falsehood of the premises." It seems to the reviewer that if Mr. Westaway is going to say as much as this he ought to say even more. At the very end of the book (p. 639) he mentions the differing points of view of Poincaré and Bertrand Russell with-

out relating it to his earlier pronouncements. The philosopher and psychologist can assert that all deduction is essentially induction. The mathematician eagerly pressing on to new discoveries can affirm his reliance on induction and intuition. But when the mathematician proceeds to an orderly arrangement of his results, two other points of view clamor for attention. One asserts that the validity of a conclusion depends upon the original premises; the other, that the validity of a set of postulates is determined by the conclusions to which they lead. It is not difficult to reconcile the first three of these four, showing intuition, induction, and deduction to be the same mental trait, differing only in degree. The fourth can be related to these three, but it also involves something essentially different from the others and this requires elaboration.

In commenting on rationalization versus habituation in arithmetic (pp. 16–20) the author asks whether a new rule ought to be explained and thoroughly understood before it is applied to examples, or whether it is legitimate to give the rule to the child dogmatically. "Suppose that we teach a rule 'intelligently' and the children get 50 per cent of their sums right; or suppose that we teach by rule of thumb and the children get 80 per cent of their sums right. Which plan should we adopt?" He concludes that we should test the results of our instruction as the modern psychologists have taught us to do; and further, that accuracy is the main thing, and no one long remembers the reasons behind certain operations. Still, he would give reasons to those who crave them. Now all this seems sensible and practical, but disappointing too. Has our instruction no ideal but utility, even in arithmetic? Granting that an objective attitude may help to rid us of prejudices, this problem is vital to all instruction and ought not to be so lightly dismissed. Does not our goal include appreciation of the methods we use? If so, and if our objective test probed also into the "why?," what then?

For example, Mr. Westaway contends (pp. 82–84) for a justification of the rule for placing the decimal point in multiplication which requires the pupil to write

$$72.314 \times .32 = \frac{72314}{1000} \times \frac{32}{100} = \frac{2314048}{100000} = 23.14048\ ;$$

but he winds up with the following observation. "Does not the time come when we *all* work mechanically in *all* types of calculation? Does not the *rationale* of procedure tend to fade away, until something turns up demanding revivification? Is there a more intelligent plan than teaching the boy to complete the actual multiplying before considering the decimal point at all? I doubt it." This appeals to the reviewer as good sense with respect to any mathematical technique, but not a good argument for insisting only on competent skill without requiring also some comprehension of the reasoning behind the technique. It ignores that as we advance in mathematics, while technique does not become less important, other matters involving reason and appreciation assume greater importance. Mr. Westaway's reference to "revivification" implies after all that

he too regards the rationale of a technique as having been born in the same generation with the technique, even though later "put away" for a period.

This question seems to the reviewer to depend upon the age of the pupil and to deserve different answers in different grades. Mr. Westaway actually takes this same stand himself, for in the case of signed numbers in algebra (p. 127) he insists on a justification of the algebraic rules derived from thermometer scales and insists that "boys who do not grasp the significance of directed numbers can never get to the bottom of their algebra; their work all through will inevitably be mechanical." The reviewer has every sympathy with the practical pedagogy which prompts this emphatic declaration, and would support it in general, but holds that in this special instance it deserves modification in view of the actual mathematics involved. Without the least detracting from the desirability of extended reference to the thermometer and other exemplifications of signed numbers, he would call attention to the mathematical fact that the rules for operations with negatives are not logically deducible from the rules for positives, but were purposely contrived to make the rules for positives apply without modification in the new field of negative numbers. The teacher may not wish to give the pupil an appreciation of this significance of negative numbers, but it would seem to the reviewer worth trying and better than relying solely on an explanation which is mathematically unsound. Mr. Westaway says (p. 129), "Teachers are not always quite happy about this question of directed numbers, and often ask if it is not unwise even to make the attempt to deal with it, and if a statement of just the rules ought not to suffice. Of the answer I have no doubt. Boys who do not grasp the significance of directed numbers can never get to the bottom of their algebra; their work all through will inevitably be mechanical." It has been the reviewer's experience that it was the brightest pupils who failed to follow the explanation of the text regarding the multiplication of -5 by -3, and that these brightest pupils were greatly relieved to discover that their inability to concur was to their very distinct credit and put them momentarily on a par with the best mathematicians.

It is too bad, of course, that the underlying theory seems at this point to support the uncomprehending use of a special technique, but there is a great difference between blindly following an unintelligible technique and employing this same technique—admittedly mechanical—with an appreciation of why it was so devised. Now Mr. Westaway is by no means ignorant of these matters, as appears from a brief paragraph with which he closes this topic (p. 132). He says, "It may be urged that the whole thing seems to be a little artificial. So it is. But the rule of signs is a universally accepted convention. The convention is perfectly self-consistent, and is easily justified, but by its nature it admits of no 'proof.'" Here at last is the other side of the question, but only hinted at. It would seem fairer to the teacher not to drop the matter with this vague reference, but to elaborate it and contrast it with the method which stresses the thermometric exemplifications. Then the teacher could decide whether to mention both aspects to his pupils, and just how he would reconcile one with the

other. For the two belong together mathematically and can be reconciled; the only compromise comes with regard to the pupil's appreciation and his psychology of learning.

The treatment of fractional exponents (p. 96) seems to need clarification. The author says, "We have learnt that $5^2 \times 5^2 = 5^{2+2} = 5^4$; so apparently we may assume that $5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$. But $\sqrt{5} \times \sqrt{5} = 5$; therefore $5^{\frac{1}{2}} = \sqrt{5}$." This procedure is not so obvious as it appears and requires amplification. The author practically defines $5^{\frac{1}{2}}$ as equal to $\sqrt{5}$; but he misses an excellent opportunity to set forth the real nature of algebra by his failure to emphasize that under the original definition of exponents in terms of positive integers the expression $5^{\frac{1}{2}}$ can have no meaning, and that our desire to generalize our algebraic symbolism prompts us to define $5^{\frac{1}{2}}$ so that the statement $5^m \times 5^n = 5^{m+n}$ shall have wider applicability.

In solving problems by means of equations, especially those involving the partition of a sum, he prefers to let x stand for the smallest number. This is good practical advice, of course; but isn't there something to be said in favor of emphasizing the idea that we can get the answer no matter how inconvenient our choice of the number to be designated x?

There is a long and excellent section devoted to graphs including a discussion of variables and functions. Later, in considering the roots of the equation $(x+2)(x-3)=0$, no mention is made of variables, although to the reviewer it would seem as though this were an opportunity not to be missed to permit x to vary continuously from infinitely large negative values to infinitely large positive values and observe what happens to the function.

Some of Mr. Westaway's suggestions on the beginning of geometry are as follows. "No attempt should be made to develop the subject on rigorously deductive lines, from first principles, though, right from the first, precise reasons for statements made should be demanded. Young boys are never happy and are often suspicious if they feel they are being asked to prove the obvious, but they can follow a fairly long chain of reasoning if the facts are clear. . . . Proofs of propositions concerning angles at a point, parallels, and congruent triangles should not be attempted, such proofs being a matter for later treatment in the upper forms. . . . By about the age of 13, a boy ought to be able to write out a simple straightforward proof formally and to attack easy riders. . . . Solid geometry of a simple kind may with great advantage be included in the early stages of any geometry course." He goes into considerable detail on all these matters, and many others, showing how to begin the deductive treatment, continuing through similar triangles to solid geometry, orthographic projection, radial projection, plane and spherical trigonometry, complex numbers, the calculus, mechanics, astronomy, optics, map projection, statistics, time and the calendar, non-Euclidean geometry, and the philosophy of mathematics. Despite the few shortcomings noted by the reviewer, it must be evident that the book covers a wide field and is a veritable store-house of valuable information for teachers of mathematics, no matter how experienced. RALPH BEATLEY

The Nine Circles of the Triangle. By W. H. Bruce. Published by the North Texas State Teachers College, 1932. 48 pages.

The nine circles treated in this booklet are the circumcircle, the inscribed and escribed circles, the nine-point circle, and the "vertical circles," whose centers are at the vertices of the triangle and which are externally tangent to each other at the points of contact of the inscribed circle and the sides. No mention is made of these three circles except in the theorem which established their existence.

The most important and best known theorems relating to the inscribed and escribed circles and the nine-point circle are established by elementary methods. The treatment is prolix, and if the reviewer reads it correctly, not always logically irreproachable. The theorem of Feuerbach is proved in two ways; one essentially that originally used by Feuerbach, the other ingenious but depending on a construction which has not been proved possible.

There is no bibliography, and not even references either to the sources or to more extensive treatments of the same field.

R. A. JOHNSON

Handbook of Mathematical Tables and Formulas. Compiled by Richard S. Burington. Sandusky, Ohio, Handbook Publishers, Inc., 1933. 252 pages. $2.00; $1.25 to students and instructors.

This excellent reference work is issued in convenient form, though rather large for the pocket. The first 86 pages are devoted to a selection of the most important formulas of mathematics from elementary algebra through the calculus, with two pages of vector analysis. There is a list of 331 integrals, together with the formulas for the usual applications of the definite integral usually studied in a first course. The standard curves of analytic geometry are reasonably well drawn, though as usual even in textbooks, the cusps of such curves as the cardioid and cycloid are not drawn with sufficient sharpness.

Most of the tables are to five places, notably the table of natural trigonometric functions;—a desirable feature with the increasing use of computing machines. There is a four-place table of trigonometric functions (and their logarithms) of angles given in radian measure. There are the usual tables of powers and roots, natural logarithms, exponential and hyperbolic functions, and their logarithms, all to five places. A number of well-chosen miscellaneous tables are followed by four-place trigonometric tables. There is an index.

The book impresses the reviewer favorably, and one anticipates that it will be widely adopted.

R. A. JOHNSON

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D.C.

All reports of club activities, suggestions and topics for club programs, and material of interest to clubs should be sent to F. M. Weida, The George Washington University, Washington, D.C. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

CLUB TOPICS

A HOME MADE MATHEMATICS EXHIBIT

By WILLIAM SELL, University of Alabama

Anyone who has seen the evident interest of a student in a cone cut into sections, or in a string model of a ruled surface, can not fail to appreciate the value of a permanent exhibit of mathematical objects.

The purpose of this note is to suggest a useful and interesting home made mathematics exhibit. It is hoped that its consideration may result in exhibits of this nature. While the exhibit suggested is far from complete, it may serve as a basis and we are anticipating that it may be a stimulus for ideas and suggestions in developing a really first class exhibit.

It will be found that practically all of the material suggested can be made by the students at little or no cost. Even facsimiles of the transit, level, compass, and sextant can be made in the woodshop and gilded to serve until the actual instruments are available. A perfunctory search through the literature will suggest other objects which may be included. The place for the exhibition is a local matter. The material should be open to the public, and available to the student for actual handling, under restrictions, otherwise much of the value will be lost.

No attempt has been made to include astronomy or mechanics. The theory of numbers has been omitted, although it has possibilities. Much of the exhibit will depend on the courses offered and on the particular likes and dislikes of those in charge of it.

Above all, it must be kept in mind that the exhibit is a growing thing, always open for additions and deletions. It must be interesting, and care will be required lest it become a dusty repository for dustier reliques. This requires that the individual in charge must be genuinely interested.

The objects that the exhibit will contain may be divided into the following six groups:

(1) Unusually excellent work by students such as solutions of difficult problems, 100% examination papers and the like; (2) Charts; (3) Models; (4) Instruments of construction; (5) Instruments of measure; (6) Historical objects.

The following outline gives a few suggestions for exhibits in Plane Geometry, Solid Geometry, Algebra, Trigonometry, Analytic Geometry, Calculus, History of Mathematics and Miscellaneous objects with reference to the above six groups.

I: *Plane Geometry:* (2) The nine-point circle; the twenty-one point circle; the principle of duality; (4) The straight edge and compass; (5) The graduated rule and other rules.

II: *Solid Geometry:* (3) Card-board models of the regular and other polyhedra; piles of closely piled spheres; the tesseract; wire and string models of various ruled surfaces.

III: *Algebra:* (2) Nomographs for roots of a quadratic and cubic equation; successive terms of a geometric series, i.e., the number of descendants of two insects by successive generations; the permutations of five letters; the circular permutations of five letters; (3) Stretched wires to show the relation of harmony and the harmonic progression; the ruled lines and needle used to find π by probability.

IV: *Trigonometry:* (2) The graphs of the trigonometric functions; a tide curve, analyzed; geometric proofs of the trigonometric identities; the value of π to 707 significant figures; (3) The spherical triangle, collapsible so as to show the derivation of the formulas.

V: *Analytic Geometry:* (2) The better known curves and surfaces with their equations; (3) Sections of a cone, sphere and other well known solids including the quadric surfaces; string, wire, paper and plaster models of well known curves and surfaces and sections of surfaces; (2) Various kinds of coordinate systems and coordinate paper.

VI: *Calculus:* (2) Infinite series for the elementary functions; "graph" of a differential equation; geometrical interpretation of the derivative and partial derivative; (5) planimeters and integrating machines.

VII: *Miscellaneous:* (2) Conformal mapping; the hare and the hound problem; Zeno's paradoxes of motion; the engine problem; the coin problem; how to tell a person's age; series of remarkable numbers; geometrical paradoxes and fallacies; the paradox of Tristram Shandy; magic squares, circles and cubes; the four-color map problem; Romeo and Juliet; mathematical puzzles; Cayley's color groups; derivation of mathematical terms; (3) Stereographic projection; Riemann surfaces; Moebius strip; quasiregular solids.

VIII: *Historical:* A collection of actual texts formerly used at this school; books written by graduates and teachers; portraits of eminent mathematicians; (2) Development of present notations; multiplication with Roman numerals.

Club Activities

1932–1933

LOCAL MATHEMATICS CLUBS

The Mathematics Club of the University of Cincinnati

The purpose of the club is to promote a deeper interest in the broader, less formal, recreational, and practical aspects of mathematics; to encourage worthy mathematical undertakings; and to afford an opportunity for social relations among the members.

All graduate and honor students in the department of mathematics are eligible for membership. Also, regular undergraduates majoring in mathematics and other qualified persons including students in the department of chemistry, physics, and engineering may become members upon invitation from the club.

Election of officers takes place at the final meeting of each semester. The officers for the first semester were: Mildren Keiffer, President; Evelyn Kennedy, Vice President; Bessie Johnson, Secretary; Alta Odoms, Treasurer. For the second semester, the officers included: Paul Pepper, President; Haim Reingold, Vice President; Stanley Klein, Secretary; Jesse Epstein, Treasurer.

The club held sixteen meetings during the year 1932–1933. The program usually consisted of the presentation of a mathematical paper and the rendition of a musical number when such could be arranged. At the conclusion of the program the members customarily enjoyed refreshments.

The meetings and programs were as follows:

October 18, 1932: Reorganization meeting.

November 1, 1932: "A new method in interpolation with finite differences" by Paul Herget.

November 18, 1932: "The International Mathematical Congress" by Dr. C. N. Moore.

December 6, 1932: "A general survey of certain mathematical publications" by Violet Diller.

December 16, 1932: Christmas tea.

January 3, 1933: "Nomograms" by Guy Harris.

January 17, 1933: "Algebraic numbers" by Dr. Harris Hancock.

February 27, 1933: "The solution of simultaneous equations by the method of iteration" by Paul Pepper.

March 9, 1933: "Topology" by Dr. William L. Ayres of the University of Michigan.

March 15, 1933: Dr. I. A. Barnett told of some of the interesting personalities which he encounterd while in Europe on his sabbatical leave.

March 30, 1933: "A method of computing trigonometric functions" by Haim Reingold.

April 10, 1933: "A simple method of computing orbits of comets" by Paul Herget.

April 26, 1933: Annual dinner. After the dinner the members journeyed to the Cincinnati Observatory where Mr. Paul Herget showed slides depicting the activities of the expedition from Cincinnati to the scene of the August, 1932, eclipse.

May 10, 1933: "Magic squares and circles" by Evelyn Kennedy.

May 19, 1933: "Topology" by Dr. John H. Roberts of Duke University.

May 24, 1933: "Divergent series" by Clemmer Mitchell.

A prize consisting of an order for a book in mathematics is being offered this year to the student who has presented at a meeting of the club the paper adjudged to be the best from the standpoint of interest to the club.

Six members of the club attended the meeting of the Ohio Section of the Mathematical Association of America at Columbus, Ohio on April 6, 1933.

EVELYN M. KENNEDY, *Chairman of the Program Committee*

The Mathematics Club of the Colleges of the City of Detroit

The mathematics club of the colleges of the City of Detroit was formed in 1925 for the purpose of stimulating a greater interest in mathematics. It is a very informal organization, its membership consisting of those students who enjoy uncovering important bypaths in the realm of mathematics.

The officers for the past year were: Margaret Dumford, President; Virginia Eyre, Secretary.

The meetings and programs were as follows:

January 12, 1933: "The division of the line segment into n equal parts" by Mr. Morris Friedman.

March 7, 1933: "Some properties of integers" by Miss Emily Taisey.

March 28, 1933: "Novel proofs of well-known theorems" by Mr. W. G. Scott.

April 11, 1933: "Mis-uses of statistics" by Miss V. L. Eyre.

VIRGINIA L. EYRE, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to Wm. Fitch Cheney, Jr., Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E60. *Proposed by M. W. Aylor, University of Virginia.*

A horse is tied by a rope to one corner of a barn which is thirty feet square. Find the length of the rope such that the horse can graze over one acre of ground without grazing over any part twice.

Editor's Note: While this type of problem is not strictly new, it is being published here as an experiment in order to learn whether or not the readers of the MONTHLY care to work on such problems.

E61. *Proposed by Raphael Robinson, University of California at Berkeley.*

Given the common logarithms of the integers from 1 to 10 correct to fifteen decimal places (as in Table Ia of the MacMillan Tables), find log 2 correct to sixteen decimal places.

E62. *Proposed by W. R. Ransom, Tufts College.*

Defining a "C-angle" as the figure formed by two internally tangent circles, and its magnitude as the difference of the curvatures of those circles, show how to bisect a C-angle geometrically. (If the circles are tangent externally, the magnitude of the C-angle is the sum of their curvatures.) If the circles are tangent to the X-axis at the origin, O, and cut the circle $x^2+y^2=2x$ also at P and Q, show that the magnitude of the C-angle equals the difference between the slopes of the chords OP and OQ.

E63. *Proposed by W. B. Carver, Cornell University.*

A number of less than thirty digits begins with the two digits 15 on the left, 15 ————; and when it is multiplied by 5, the result is merely to move these two digits to the right-hand end, thus, ———— 15. Find the number, and show that the solution is unique.

E64. *Proposed by J. Rosenbaum, The Milford School, Milford, Connecticut.*

The bisectors of the interior angles of the triangle ABC meet the sides in the points P, Q and R. Prove that the ratio of the area of the triangle PQR to the area of the triangle ABC is $2abc/(a+b)(b+c)(c+a)$.

SOLUTIONS

E13. [1932, 606]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

It is required to find two factors which, together with their product, contain each of the nine digits from one to nine just once. Each of the several solutions should be found. (Zero must not appear in any solution.)

Solution by W. E. Buker, Leetsdale, Pa.

The following multiplications satisfy the requirements of the problem

$4 \times 1738 = 6952$	$4 \times 1963 = 7852$	$12 \times 483 = 5796$
$18 \times 297 = 5346$	$27 \times 198 = 5346$	$28 \times 157 = 4396$
$39 \times 186 = 7254$	$42 \times 138 = 5796$	$48 \times 159 = 7632.$

Can anyone prove that these are the only solutions, or find another?

E31. [1933, 241]. *Proposed by W. R. Ransom, Tufts College.*

Point P is on the line AB, one N-th of the way from A to B, and Q is on the line AC, one N-th of the way from A to C. PC intersects QB at R. Prove that the area of the triangle ABC is $N(N+1)/2$ times the area of the quadrilateral $APRQ$.

Solution by Roy MacKay, Albuquerque, New Mexico.

Denote the area of the triangle ABC by a and that of the quadrilateral $APRQ$ by q. Then since the triangles ABC and CRB are respectively similar to the triangles APQ and PRQ with the ratio of similitude equal to N in each case, the area of the triangle $APQ = a/N^2$ and that of triangle $CBR = N^2(q - a/N^2)$. Furthermore, the sum of the areas of the triangles APC and ABQ is $2a/N$. Hence $a = 2a/N - q + N^2(q - a/N^2)$. When solved for a, this gives the desired result: $a = [N(N+1)/2]q$.

Solved also by W. E. Buker, J. H. Butchart, Wm. Douglas, F. C. Gentry, C. W. Munshower, C. T. Oergel, Simon Vatriquant, R. N. Walter and R. C. Yates.

E32. [1933, 241]. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Prove that $\int \sec x\, dx = -2\, i \tan^{-1} e^{ix} + C$, where $i = \sqrt{-1}$.

Solution by Mary Hamilton, Agnes Scott College, Decatur, Georgia.

$$\int \sec x\, dx = \int dx/\cos x = 2\int dx/(e^{ix} + e^{-ix}) = (2/i)\int ie^{ix}dx/(e^{2ix} + 1)$$

$$= (2/i)\tan^{-1} e^{ix} + C = -2i \tan e^{ix} + C.$$

Solved also by Clyde Bridger, J. H. Butchart, F. C. Gentry, Roy MacKay, F. L. Manning, J. A. McLaughlin, Lazarus Medveson, Jr., W. R. Ransom, C. C. Richtmeyer, Rafael Sanchez-Diaz, Simon Vatriquant and R. C. Yates.

E33. [1933, 241]. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

The points A and B are any two points not in the plane M. Find the locus of the point X, in M, such that the lines AX and BX make equal angles with M.

Solution by J. Rosenbaum, The Milford School, Milford, Connecticut.

Let the projections of A and B on M be A' and B'. Then by the condition of the problem the right triangles $AA'X$ and $BB'X$ are similar. Hence $XA'/XB' = AA'/BB' =$ a constant. From this it follows that the locus of X is an Apollonian circle. See *College Geometry* by N. A. Court, page 14, Locus 11.

Note: It is seen that the above solution holds whether A and B are on the same side or on opposite sides of M. It is also seen that when A and B are equidistant from M, the locus becomes a straight line which is the perpendicular bisector of $A'B'$.

Solved also by J. H. Butchart, F. C. Gentry, Roy MacKay, Lazarus Medveson, Jr., W. R. Ransom, Simon Vatriquant and R. N. Walter.

E35. [1933, 241]. *Proposed by W. B. Campbell, Rangoon, Burma.*

Show that, under appropriate conditions, the limit as n approaches minus one, of the integral $\int_a^b X^n dX$ is equal to the integral $\int_a^b X^{-1} dX$.

Solution by Lazarus Medveson, Jr., Albuquerque, New Mexico.

$$\lim_{n \to -1} \int_a^b X^n dX = \lim_{n \to -1} \frac{b^{n+1} - a^{n+1}}{n+1}.$$

If both a and b are finite and neither is zero, the above fraction assumes the indeterminate form $0/0$ when $n = -1$. In such cases the limit of the value of the fraction as $n \to -1$ is

$$\lim_{n \to -1} \frac{D_n(b^{n+1} - a^{n+1})}{D_n(n+1)} = \lim_{n \to -1} \frac{b^{n+1} \log b - a^{n+1} \log a}{1}$$

which equals $\log b - \log a$, which is also $\int_a^b X^{-1} dX$.

Editor's Note: A further necessary condition is that both limits of integration have the same algebraic sign, as the first step in the above solution is not valid otherwise.

Solved also by E. F. Allen, W. R. Ransom, J. Rosenbaum and Simon Vatriquant.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In

general, problems in well-known textbooks or results found in readily accessible sources will not, be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3639. *Proposed by N. A. Court, University of Oklahoma.*

The two planes passing through the circumcenter of a tetrahedron and perpendicular to two bimedians (i.e., lines joining the mid-points of pairs of opposite edges) divide the third bimedian harmonically.

3640. *Proposed by V. Thébault, Le Mans, France.*

Let $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ be the pedal triangles of the two isogonally conjugate points P, P' with respect to the triangle ABC. The parallels through P, P' to the sides of the triangles $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$ meet the sides BC, CA, AB, in two sets of three collinear points. These two lines are respectively perpendicular to the lines joining P', P to the orthocenters of the triangles $\alpha'\beta'\gamma'$, $\alpha\beta\gamma$.

3641. *Proposed by J. A. Bullard, University of Vermont.*

Prove, for positive integers p and q, the following summations of binomial coefficients:

(a) $$\sum_{h=0}^{p} \frac{(-1)^h}{2h+1} {}_pC_h = 2^{2p}(p!)^2/(2p+1)!$$

(b) $$\sum_{h=0}^{p+q} (-1)^h {}_{2p}C_h \, {}_{2q}C_{p+q-h} = (-1)^p(2p)!(2q)!/(p+q)!p!q!$$

If $p \neq q$ in (b) the terms which are undefined are to be omitted in the summation.

3642. *Proposed by H. Halperin, A. and M. College of Texas.*

Show that the caustic curve by reflection from a circle, of the rays issuing from a point on the circle, can be generated as follows:

Let A be the point source on the circle O; B a point on the diameter AOB, such that OB equals one third of AO; M a variable point on the circle. Then the point P of intersection of the reflection MP of the ray AM with the line BP parallel to OM generates the caustic curve.

3643. *Proposed by H. G. Green, University College, Nottingham, England.*

Show that, if a and b are positive numbers with b greater than a, and $e\xi = (b^b/a^a)^{1/(b-a)}$, then ξ lies between a and b. Given that $\log_{10} e = 0.4343$ to four decimal places, deduce that $\log_{10} 99$ lies between 1.99565750 and 1.99561262, and explain theoretically the close value of the mean of these numbers to the true value of the log.

3644. *Proposed by J. Rosenbaum, the Milford School, Milford, Connecticut.*

Prove that in a tetrahedron, the three conditions:

1. The altitudes are concurrent,
2. The sums of the squares of the pairs of opposite edges are equal, and
3. The opposite edges are perpendicular to each other,

are equivalent (i.e. anyone of the conditions implies the other two).

3645. *Proposed by Paul S. Dwyer, Antioch College.*

Show that the value of the determinant formed by deleting the kth column from the array

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & {}_1C_1 & {}_2C_1 & {}_3C_1 & {}_4C_1 & \cdots & {}_nC_1 & {}_{n+1}C_1 \\ 0 & 0 & {}_2C_2 & {}_3C_2 & {}_4C_2 & \cdots & {}_nC_2 & {}_{n+1}C_2 \\ 0 & 0 & 0 & {}_3C_3 & {}_4C_3 & \cdots & {}_nC_3 & {}_{n+1}C_3 \\ 0 & 0 & 0 & 0 & {}_4C_4 & \cdots & {}_nC_4 & {}_{n+1}C_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & {}_nC_n & {}_{n+1}C_n \end{Vmatrix}$$

is ${}_nC_{k-1}$.

SOLUTIONS

3536. [1932, 175]. *Proposed by Martin Rosenman, Brooklyn, N. Y.*

Consider fractions of the form $1/2, 1/3, 1/4, 1/5, \cdots$. We seek to determine which n of these fractions (repetitions allowed) give a sum as near unity as possible but actually less than it. Thus for $n=3$, we have $1/2+1/3+1/7 = 41/42$. Prove or disprove that, in general, the first n of the fractions in the series $1/2+1/3+1/7+1/43+1/1807 \cdots$ give the desired result; in which series each denominator exceeds by 1 the product of all preceding.

Solution by D. R. Curtiss, Northwestern University.

From a communication from Professor Curtiss it has been learned by the editors that this problem is essentially the one considered by O. D. Kellogg in his paper, *On a Diophantine problem*, in this MONTHLY, vol. 28 (1921), p. 300; and that a complete solution of this problem was given by Curtiss in his paper, *On Kellogg's Diophantine problem*, in this MONTHLY, vol. 29 (1922), p. 380. The editors regret that through oversight the earlier form of the problem and its solution were overlooked at the times of the printing of problem 3536 and of the partial solution [1933, 180].

3582. [1932, 607]. *Proposed by Dewey C. Duncan, University of California.*

If α and β are positive integers and $\beta>2$, then $2^\alpha+1$ is never divisible by $2^\beta-1$.

Solution by C. H. Smiley, Brown University.

First we note that if $\beta>2$, $2^{\beta-1}(2-1)>2$, that is, $2^{\beta-1}+1<2^\beta-1$. Hence

(1) if $\alpha<\beta$, $\quad 2^\alpha+1<2^\beta-1$,

and $2^\alpha+1$ is clearly not divisible by $2^\beta-1$.

Next, if $\alpha=\beta$, $(2^\alpha+1)/(2^\beta-1)=1+2/(2^\beta-1)$ and again, $2^\alpha+1$ is not divisible by $2^\beta-1$.

Finally suppose $\alpha>\beta$, say $\alpha=m\beta+n$ where m is a positive integer and n is zero or a positive integer less than β. Then

$$\frac{2^\alpha+1}{2^\beta-1}=\frac{2^\alpha-2^{\alpha-m\beta}}{2^\beta-1}+\frac{2^n+1}{2^\beta-1}.$$

The numerator of the first fraction on the right may be written $2^{\alpha-m\beta}(2^{m\beta}-1)$ which is clearly divisible by $2^\beta-1$; and by (1), the second fraction on the right is a proper fraction. Thus, for all cases, including $\alpha>\beta$, we have shown that $2^\alpha+1$ is not divisible by $2^\beta-1$ when $\beta>2$.

Solved also by A. G. Clark, Bernard Friedman, R. Graves, H. G. Green, Ruth G. Mason, L. C. Mathewson, A. Pelletier, J. Rosenbaum, S. Vatriquant, and the proposer.

3584. [1932, 608]. *Proposed by W. E. Buker, Leetsdale, Pa.*

Find the rational values of x and y for which

$$x^3+y^3=2^3+1^3.$$

Partial Solution by A. Pelletier, Montreal, Canada.

Set $y=kx$, then

$$x^3=\frac{27}{3+3k^3}; \tag{1}$$

and we have to make the denominator a cube. We know that it is a cube for $k=2$ or $1/2$; and in order to find other values of k, we set $k=2+m$, and then find m so that the denominator is a cube. Thus

$$\begin{aligned} 3+3k^3 &= 27+36m+18m^2+3m^3, \\ &= (3+am)^3=27+27am+9a^2m^2+a^3m^3. \end{aligned} \tag{2}$$

If we take for a the value $4/3$ we then find $m=-54/17$, and then

$$k=-\frac{20}{17}, \qquad x=-\frac{17}{7}, \qquad y=\frac{20}{7}.$$

Again, setting $k=-20/17+m$ and proceeding as before, we get another solution, and so on. In this manner we see that there are an infinite number solutions.

Note by the Editors. It is not proved in this solution that this process gives all rational values of x and y satisfying the given equation.

3586. [1932, 608]. *Proposed by R. E. Gaines, University of Richmond.*

If while an ellipse is turned about in its plane it remains tangent to a fixed straight line at a fixed point, its foci trace a curve whose area is $2\pi a(a-b)$.

Solution by Eugene M. Berry, Lynchburg College.

In terms of the eccentric angle, t, the equations for the ellipse are $x=a\cos t$, $y=b\sin t$, where the x-axis is the major axis and the y-axis is the minor axis of the ellipse. Let (x', y') be the coordinates of the focus referred to the fixed straight line as x'-axis and the fixed point on this line as origin. Then y' is the distance from the focus $(-c, 0)$ to the tangent line and x' the distance from $(-c, 0)$ to the normal line.

The tangent and normal lines are

$$xb\cos t + ya\sin t - ab = 0,$$
$$xa\sin t - yb\cos t - c^2\sin t\cos t = 0,$$

where $c^2=a^2-b^2$. Hence

$$y' = b(a + c\cos t)^{1/2}(a - c\cos t)^{-1/2},$$
$$x' = c\sin t(a + c\cos t)^{1/2}(a - c\cos t)^{-1/2}.$$

Since the curve is symmetrical to the y'-axis, the area K is given by

$$K = \int_{t=\pi}^{t=0} x'dy' = 2abc^2\int_0^{\pi}\sin^2 t(a - c\cos t)^{-2}dt.$$

Integrating once by parts and putting in the limits we get

$$K = 2abc\int_0^{\pi}\cos t(a - c\cos t)^{-1}dt$$
$$= -2ab\int_0^{\pi}dt + 2a^2b\int^{\pi}(a - c\cos t)^{-1}dt.$$

Hence

$$K = -2abt + 4a^2b(a^2 - c^2)^{-1/2}\arctan\left\{(a + c)^{1/2}(a - c)^{-1/2}\tan\tfrac{1}{2}t\right\}\Big]_0^{\pi}$$
$$= 2\pi a(a - b).$$

If in the above equations, we replace $-c$ by c and t by $\pi+t$, the equations are unchanged. From this we see that the curve is the same whichever focus is used.

Solved also by H. G. Green, William Hoover, A. Pelletier, and F. Underwood.

A Note by Otto Dunkel. The form of the locus may be easily seen from its equation in polar coordinates. If O is the fixed point and F and F' are the two foci, we take as the polar axis the bisector of angle $FOF'=2\theta$ cutting FF' in N. Setting $OF=\rho$, $OF'=\rho'$, we have by the Law of Cosines

$$\rho\rho' = b^2\sec^2\theta, \qquad \rho + \rho' = 2a, \qquad \rho^2 - 2a\rho + b^2\sec^2\theta = 0.$$

This last equation shows the oval form of the curve. Its area A is given by

$$A = 4a \int_0^{\cos^{-1}(b/a)} (a^2 - b^2 \sec^2 \theta)^{1/2} d\theta.$$

Setting ϕ = angle ONF', we have $\sin \theta = e \sin \phi$, and with this relation the above integral becomes

$$A = 4c^2 \int_0^{\pi/2} \frac{\cos^2 \phi d\phi}{1 - e^2 \sin^2 \phi}.$$

In order to show that this gives the desired result, we consider the polar equation of an ellipse with the polar axis along the minor axis and with the pole at the center,

$$\rho^2 = \frac{b^2}{1 - e^2 \sin^2 \phi}.$$

The area between the auxiliary circle and the ellipse is then

$$B = 2c^2 \int_0^{\pi/2} \frac{\cos^2 \phi d\phi}{1 - e^2 \sin^2 \phi}.$$

Hence $A = 2B$, which is the result to be proved. From the simplicity of the result one would expect that it would be possible to derive it by geometrical methods.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

Contracts for the new McDonald Observatory of the University of Texas have been signed. The Observatory will have an eighty-inch reflecting telescope. The big mirror will be made of pyrex glass with a low coefficient of expansion. The Observatory will be located on Mount Locke, a seven thousand foot peak of the Davis Range in southwestern Texas. Dr. Otto Struve, director of the Yerkes Observatory of the University of Chicago, will also be director of the McDonald Observatory, under a joint agreement of the two universities. It is expected that the new telescope will be ready in two years. It is designed to gather large amounts of light and will be used for photographing faint stars in certain constellations.

In order to carry out the full intent of the National Research Council, President Roosevelt, by an executive order, has created a Science Advisory Board with authority. This board acts through the machinery and under the jurisdiction of the National Academy of Sciences and the National Research Council to appoint committees to deal with specific problems in the various departments. The Advisory Board consists of nine members appointed for a term of two years. Karl T. Compton, President of the Massachusetts Institute of Technology, and Professor R. A. Millikan, of the California Institute of Technology, have been appointed as members of the Advisory Board.

Professor W. D. Cairns of Oberlin College gave an address before the mathematics section of the Western Pennsylvania Educational Conference at Pittsburgh, October 7, 1933, on the subject "The fundamental theorem of algebra—an exposition."

Professor R. D. Carmichael, head of the department of mathematics at the University of Illinois, has been made acting dean of the Graduate School in that institution.

At the summer meeting of the North Carolina Academy of Sciences Professor E. L. Mackie of the University of North Carolina was elected chairman, and Professor E. R. C. Miles of Duke University was elected secretary of the mathematics section.

Professor E. G. Bill, who for many years has been director of admissions and dean of freshmen at Dartmouth College, has recently been made dean of the faculty at that institution. More recently, Dean Bill has been appointed by President Roosevelt to a federal commission which will effect the amalgamation of the Bureaus of Immigration and Naturalization.

Dr. L. E. Bush, instructor in mathematics at the Ohio State University, has been appointed professor of mathematics at the College of St. Thomas, St. Paul, Minnesota.

Professor L. L. Dines of the University of Saskatchewan has been appointed professor of mathematics and head of the department at the Carnegie Institute of Technology.

Mr. F. F. Middleswart has been appointed professor of mathematics at Alderson-Broaddus Junior College, Philippi, West Virginia.

Dr. A. L. O'Toole has been appointed to a professorship in mathematics at Mary Manse College, Toledo, Ohio.

Dr. J. H. Butchart has been appointed instructor of mathematics at Butler University.

Dr. J. L. Dorroh has been appointed to an instructorship in mathematics at Johns Hopkins University.

Dr. M. H. Martin, National Research Fellow at Harvard University, has been appointed instructor in mathematics at Trinity College.

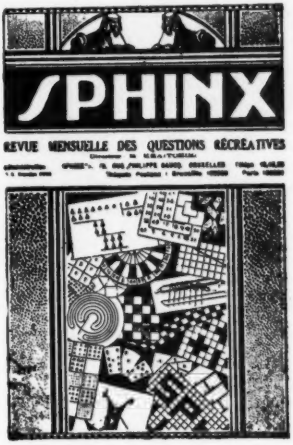
National Research Fellowships in Mathematics for the year 1933–1934, were awarded to the following:

Reappointed for a second year

Doob, J. L.	Martin, M. H.	Titt, E. W.
James, R. D.	Martin, R. S.	

First year appointments

Barber, S. F.	Hestenes, M. R.	Montgomery, Deane
Blumenthal, L. M.	Hull, Ralph	Nathan, D. S.
Cameron, R. H.	Lewis, D. C., Jr.	Sullivan, M. M.
Hedlund, G. A.	Lorch, E. R.	



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The Chauvenet Prize

In the year 1925, the Mathematical Association of America established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the Carus Monographs are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the Chauvenet Prize will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1932, to Professor G. H. Hardy. The next award will be in December, 1935, for the period 1931-1934.

Note that the prize is to be awarded only to a member of the Association—one more of the many good reasons for membership.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

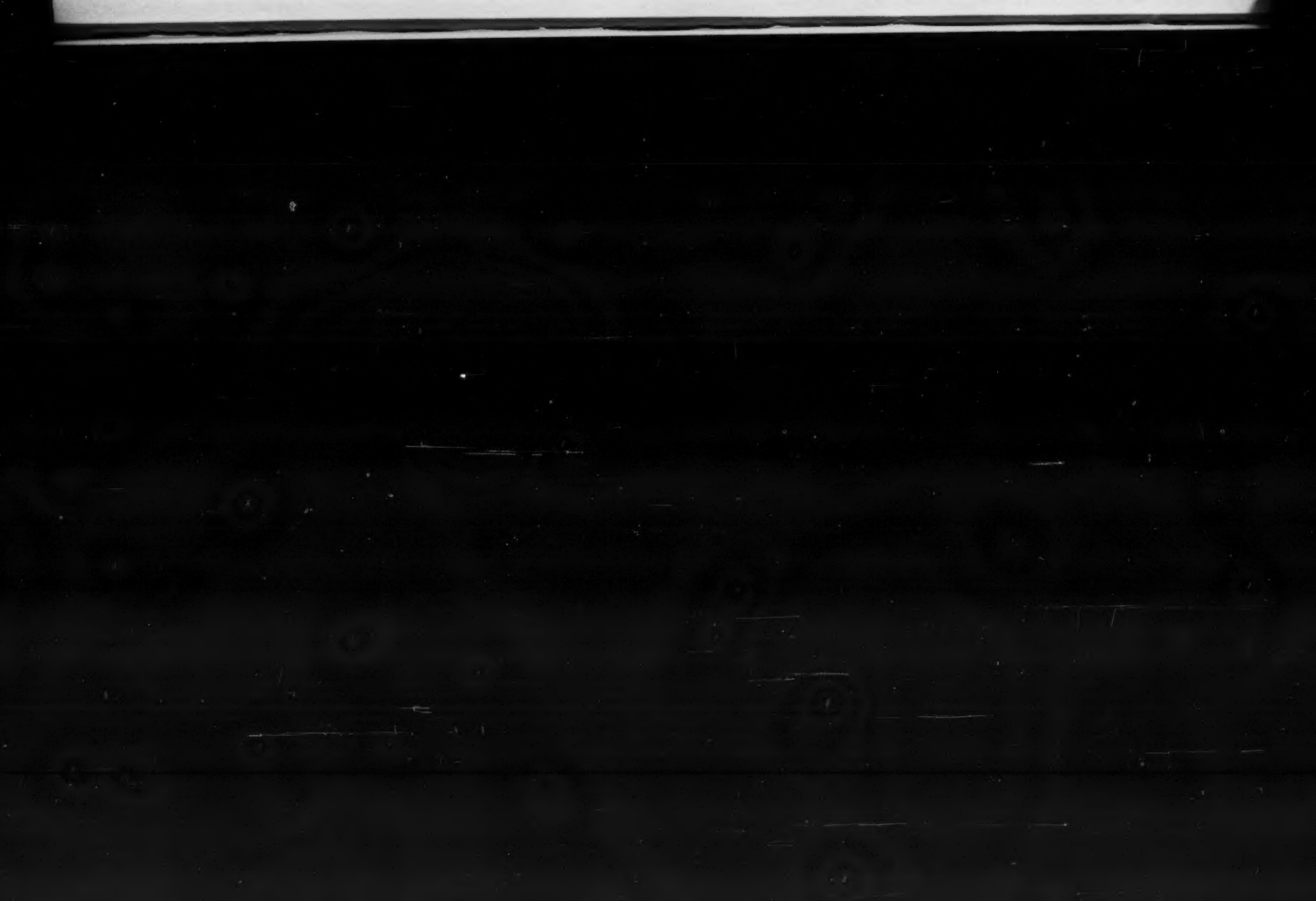
MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Eighteenth Annual Meeting of the Association, Cambridge, Mass., Dec. 27-29, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS, merged with the Chicago meeting.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13; Washington, D.C., Dec. 9.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.



THE APRIL MEETING OF THE WISCONSIN SECTION

The first annual meeting of the Wisconsin Section was held at Beloit College, Beloit, Wisconsin, on Saturday, April 8, 1933.

The two sessions, morning and afternoon, were presided over by the chairman, Professor G. A. Parkinson, of the University of Wisconsin Extension Division, Milwaukee.

The attendance was forty-three, including the following eighteen members of the Association; Florence E. Allen, Leon Battig, May M. Beenken, H. H. Conwell, L. A. V. DeCleene, W. W. Hart, M. L. Hartung, R. C. Huffer, M. H. Ingraham, R. E. Langer, Morris Marden, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, H. E. Slaught, I. S. Sokolnikoff, J. I. Vass.

Officers elected for the year 1933–34 were: Chairman, G. A. Parkinson; Secretary, H. P. Pettit; Program Committee, Mary B. McMillan, H. H. Conwell. The fall meeting was scheduled for the first part of November, definite arrangements to be left in the hands of the program committee. The second annual meeting was set for Saturday May 5, 1934, at Oshkosh State Teachers College.

The following five papers were read:

1. "Improving the teaching of college mathematics" by Professor May M. Beenken, Oskhosh State Teachers College.

2. "Some general remarks concerning scientific exposition" by Professor R. E. Langer, University of Wisconsin.

3. "Singularities of certain algebraic plane curves" by Professor H. P. Pettit, Marquette University.

4. "Reminiscences of Professor E. H. Moore" by Professor M. H. Ingraham, University of Wisconsin.

5. "The lag of mathematics behind literature and art in the earlier centuries" by Professor H. E. Slaught.

Abstracts of two of the papers follow:

1. Teaching is the chief function of the undergraduate college. Research is of importance as a reinforcement of teaching. The successful mathematics teacher must not only be a master of his subject with a broad knowledge of present day applications of mathematics, but he must have a genuine interest in the students he is teaching. College teaching of mathematics can be improved by giving due recognition to excellence in teaching and by providing the Ph.D. candidate with definite training for his teaching profession.

3. Professor Pettit in a paper entitled "Projective description of some higher plane curves," The Tôhoku Mathematical Journal, 1927, dealt with the construction of a curve of order $2mn$ by means of the correspondence set up between two pencils of lines with base curves of order m and n, respectively. This gives the usual projective description of the conic for $m=n=1$. The existence of m n-fold and of n m-fold points was shown. In the present paper the